TENSOR PRODUCTS OF FELL BUNDLES OVER DISCRETE GROUPS

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Abstract. We prove, by means of the tensor product of Fell bundles, that a Fell bundle $\mathcal{B} = \{B_t\}_{t\in G}$ over a discrete group G has nuclear cross-sectional C^*-algebra, whenever $\mathcal{B}$ has the approximation property and the unit fiber B_e is nuclear. In particular, any twisted partial crossed product of a nuclear C^*-algebra by an amenable discrete group is nuclear.

1. Introduction

Amenability of Fell bundles was introduced by Exel in [4] for Fell bundles over discrete groups. He defined a reduced cross-sectional algebra $C_r^*(\mathcal{B})$ for a Fell bundle $\mathcal{B}$, that generalizes the reduced C^*-algebra of a group and reduced crossed products. As in these cases, one has a natural epimorphism $\Lambda : C^*(\mathcal{B})\longrightarrow C_r^*(\mathcal{B})$, called the left regular representation (here $C^*(\mathcal{B})$ is the full cross-sectional algebra of the Fell bundle $\mathcal{B}$). When the left regular representation is injective, that is, an isomorphism, the Fell bundle is called amenable. For example, a discrete group G produces a Fell bundle $\mathcal{B}_G$ with constant fiber $\mathbb{C}$, called the *group bundle of* G, and in this case we have $C^*(\mathcal{B}_G) = C^*(G)$, $C_r^*(\mathcal{B}_G) = C_r^*(G)$; so $\mathcal{B}_G$ is an amenable Fell bundle if and only if G is an amenable group. But it may be the case of a Fell bundle to be amenable even if the group is non-amenable. In [4], an important example of this situation is treated: a Fell bundle over the free group on n generators $\mathbb{F}_n$, whose cross-sectional algebra is the C^*-algebra of Cuntz–Krieger $\mathcal{O}_A$. Although $\mathbb{F}_n$ is a non-amenable group, this bundle is shown to be amenable, because it satisfies the *approximation property* (see Definition 4.1). The concept of a Fell bundle with the approximation property was also introduced in [4], where it is shown that if $\mathcal{B}$ has the approximation property, then $\mathcal{B}$ is amenable, and if $\mathcal{B}$ is a Fell bundle over an amenable group, then $\mathcal{B}$ has the approximation property, and hence is amenable. Later, Exel studied the amenability of Fell bundles over free groups, and proved that if such a bundle is orthogonal, semi-saturated and separable, then it is amenable (see [5]). In a recent work ([6]), Exel and Laca introduced a generalization of the Cuntz–Krieger algebras to the case of an arbitrary matrix of "zeros" and "ones" without null rows, and proved that it is a crossed product of a commutative C^*-algebra by a partial action of a free group. Exel also showed in [5] that the corresponding associated Fell bundle has the approximation property. So, we see that these generalized Cuntz–Krieger algebras are nuclear by applying our main result (see [6] for a complete information).

In this paper we will be concerned with tensor products of Fell bundles over discrete groups, introduced in section 3, and its relation with amenability. If $\mathcal{A}$, $\mathcal{B}$ are Fell bundles over discrete groups G and H, then $\mathcal{A}\bigotimes_\alpha\mathcal{B}$ will be a Fell bundle over $G\times H$, and the general idea is to compare C^*-algebras associated to $\mathcal{A}\bigotimes_\alpha\mathcal{B}$, like $C^*(\mathcal{A}\bigotimes_\alpha\mathcal{B})$ and $C_r^*(\mathcal{A}\bigotimes_\alpha\mathcal{B})$, with tensor products of C^*-algebras associated to $\mathcal{A}$ and $\mathcal{B}$, like $C^*(\mathcal{A})\bigotimes_{\max}C^*(\mathcal{B})$ and $C_r^*(\mathcal{A})\bigotimes_{\min}C_r^*(\mathcal{B})$. The crucial result is the diagram D_α of Theorem 4.11, which allows us to prove that a Fell bundle with the approximation property and nuclear unit fiber has a nuclear cross-sectional algebra.

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Let us briefly describe the organization of the paper.

In the next section we study tensor products of C^*-ternary rings, as the basic step for dealing later with tensor products of Fell bundles. The main result of this section is Theorem 2.25, where it is shown that if E, F are full right Hilbert modules over C^*-algebras B_E and B_F, then there is a natural bijective correspondence between the set of tensor products $B_E \bigotimes B_F$ and the set of tensor products $E \bigotimes F$. As a consequence we obtain that there exists an embedding $\mathcal{L}(E) \bigotimes \mathcal{L}(F) \hookrightarrow \mathcal{L}(E \bigotimes F)$, where $\mathcal{L}(E)$ means the C^*-algebra of adjointable maps on E (this generalizes the well known fact that $B(H) \bigotimes_{\min} B(K) \hookrightarrow B(H \bigotimes K)$, for Hilbert spaces H and K). This result will be useful later for proving our main result.

The third section is devoted to the definition and construction of tensor products of Fell bundles over discrete groups. It is shown that there exist a maximal and a minimal tensor products, such that their unit fibers are, respectively, the maximal and minimal tensor products of the unit fibers of the Fell bundles tensorized.

In the final section, we begin by showing that if $\mathcal{A}$ and $\mathcal{B}$ are Fell bundles with the approximation property, then $\mathcal{A} \bigotimes \mathcal{B}$ also satisfies the approximation property, for each possible tensor product. After that we work to obtain the diagrams D_α of Theorem 4.11, the main result of the paper. We then deduce that a Fell bundle with the approximation property and nuclear unit fiber has nuclear cross-sectional algebra (4.13). In particular, any twisted partial crossed product of a nuclear C^*-algebra by an amenable group is nuclear.

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2. Tensor Products of C^*-Ternary Rings

In this section we will define tensor products of C^*-ternary rings, as a preliminary step before dealing with tensor products of Fell bundles. We begin with some algebraic facts, after what we recall the definition and main results about C^*-ternary rings. Next, we begin the study of tensor products of C^*-ternary rings, specially the positive ones (2.15), that are essencially full Hilbert modules (see Definition 2.15, and Remarks 2.14 and 2.16). Every positive C^*-ternary ring E has associated a C^*-algebra B_E, in a way that E is a full right Hilbert B_E-module. Then, generalizing the *exterior tensor product of Hilbert modules* (see for instance [8]), we see that every tensor product $B_E \bigotimes_\gamma B_F$ induces a tensor product $E \bigotimes_{\tilde{\gamma}} F$ of the positive C^*-ternary rings E and F (Proposition 2.21). Conversely, we show in Proposition 2.24, that every tensor product $E \bigotimes_\alpha F$ defines a tensor product $B_E \bigotimes_{\overline{\alpha}} B_F$ of the respective associated algebras. These correspondences are shown to be bijective and mutually inverse in Theorem 2.25, the main result of the section. As a Corollary, we obtain that there are a maximal tensor product $E \bigotimes_{\max} F$ and a minimal tensor product $E \bigotimes_{\min} F$, and that their associated algebras are respectively $B_E \bigotimes_{\max} B_F$ and $B_E \bigotimes_{\min} B_F$. The section is ended with an application of the results above, by showing a similar "$B(H) \bigotimes_{\min} B(K) \subseteq B(H \bigotimes K)$" result, that will be useful later in the last section.

Definition 2.1. A *-ternary ring is a complex vector space E with a map $\mu : E \times E \times E \longrightarrow E$, called a *-ternary product on E, which is linear in the odd variables and conjugate linear in the second one, and such that:

$$\mu(\mu(x, y, z), u, v) = \mu(x, \mu(u, z, y), v) = \mu(x, y, \mu(z, u, v)),\ \forall x, y, z, u, v \in E$$

A morphism $\phi : (E,\mu) \longrightarrow (F,\nu)$ of *-ternary rings is a linear map such that

$$\nu(\phi(x),\phi(y),\phi(z)) = \phi(\mu(x,y,z)),\ \forall x,y,z \in E.$$

When no confusion may occur, we will write just (x,y,z) instead of $\mu(x,y,z)$. We will use the abbreviated expression *-tring instead of *-ternary ring.

Remark 2.2. We are calling here *-trings to the same objects that appear in [9] under the name "associative triple systems of the second kind". We think our terminology is more coherent with the notion of C^*-ternary rings due to Zettl (Definition 0.1 of [11]), because a C^*-ternary ring is a completion of a *-tring under a suitable norm (see Definition 2.10).

Example 2.3. Let $E = M_{m\times n}(\mathbb{C})$ be the space of all $m \times n$ matrices with entries in $\mathbb{C}$, and let $\mu(x,y,z) = x.y^*.z$, where y^* is the conjugate transpose matrix of y, and $x.y$ means the usual matrix product. Then, it easy to check that (E,μ) is a *-ternary ring.

Example 2.4. Suppose (E,μ) is a *-ternary ring. Then so is $(E,-\mu)$, as it is easy to check. We say that $(E,-\mu)$ is the anti-*-ternary ring of (E,μ). If ϕ is a morphism from a *-ternary ring E to the anti-*-ternary ring of F, we say that ϕ is an anti-morphism from E to F, and that E and F are anti-isomorphic in case ϕ is an isomorphism.

Example 2.5. If E is a given complex vector space, let us denote by E^* the complex vector space which is equal to E as an abelian group, and where the $\mathbb{C}$-action is given by $\lambda.x = \bar{\lambda}x$, $\forall x \in E, \lambda \in \mathbb{C}$, and where the right term of the equality is the original action of $\mathbb{C}$ on E. Now, if (E,μ) is a *-ternary ring, then $\mu^* : E^* \times E^* \times E^* \longrightarrow E^*$ given by $\mu^*(x,y,z) = \mu(z,y,x)$, $\forall x,y,z \in E^*$, is a *-ternary product on E^*. We call the *-tring (E^*,μ^*) the *adjoint *-ternary ring* of (E,μ).

In the next Lemma, we denote the algebraic tensor product of the spaces $E_1,\dots,E_n$ by $E_1 \bigodot \dots \bigodot E_n$, or just by $\bigodot_{j=1}^n E_j$. We will use this notation along all the text.

Lemma 2.6. *Let E_{ij}, F_i be complex vector spaces, $\forall i = 1,\dots,m$, $j = 1,\dots,n$, and suppose we are given n-linear maps $\alpha_i : \prod_{j=1}^n E_{ij} \longrightarrow F_i$, for each $i = 1,\dots m$. Then there exists a unique n-linear map $\alpha := \alpha_1 \odot \cdots \odot \alpha_m$,*

$$\alpha : \prod_{j=1}^{n} \bigodot_{i=1}^{m} E_{ij} \longrightarrow \bigodot_{i=1}^{m} F_i$$

such that $\alpha(\odot_{i=1}^m e_{i1},\dots,\odot_{i=1}^m e_{in}) \mapsto \odot_{n=1}^m \alpha_i(e_{i1},\dots,e_{in})$

Proof. The existence follows from successive use of the natural isomorphisms $L(V_1,\dots V_n;W) \cong L(\bigodot_{j=1}^t V_j, W)$, where $L(V_1,V_2,\dots V_n;W)$ denotes the linear space of W-valued n-linear maps on $V_1 \times \dots V_n$. The uniqueness is evident, because the map α has prescribed values on a generating set of its domain. $\square$

Proposition 2.7. *If (E,μ), (F,ν) are *-trings, then $(E \bigodot F, \mu \odot \nu)$ is a *-tring.*

Proof. Let E^* be the conjugate space of E, as in example 2.5. Because the maps

$$\mu : E \times E^* \times E \longrightarrow E \qquad \text{and} \qquad \nu : F \times F^* \times F \longrightarrow F$$

are trilinear, so is $\mu \odot \nu : (E \bigodot F) \times (E^* \bigodot F^*) \times (E \bigodot F) \longrightarrow E \bigodot F$. But $E^* \bigodot F^* = (E \bigodot F)^*$, so $\mu \odot \nu : (E \bigodot F) \times (E \bigodot F)^* \times (E \bigodot F) \longrightarrow E \bigodot F$ is a trilinear map and, for checking the algebraic identities of the definition on elementary tensors, let $(\star)$ be:

$$(\star) \quad = \quad \mu \odot \nu(\mu \odot \nu(x_1 \odot y_1, x_2 \odot y_2, x_3 \odot y_3), x_4 \odot y_4, x_5 \odot y_5)$$

Then:

$$\begin{aligned}
(\star) &= \mu \odot \nu(\mu(x_1, x_2, x_3) \odot \nu(y_1, y_2, y_3), x_4 \odot y_4, x_5 \odot y_5) \\
&= \mu(\mu(x_1, x_2, x_3), x_4, x_5) \odot \nu(\nu(y_1, y_2, y_3), y_4, y_5) \\
&= \mu((x_1, \mu(x_4, x_3, x_2), x_5) \odot \nu(y_1, \nu(y_4, y_3, y_2), y_5) \\
&= \mu \odot \nu((x_1, \mu(x_4, x_3, x_2), x_5) \odot (y_1, \nu(y_4, y_3, y_2), y_5)) \\
&= \mu \odot \nu(x_1 \odot y_1, \mu \odot \nu(x_4 \odot y_4, x_3 \odot y_3, x_2 \odot y_2), x_5 \odot y_5)
\end{aligned}$$

The other identity is verified with analogous calculations. □

Definition 2.8. A ternary ring of operators is a complex subspace E of $B(H, K)$, where H, K are complex Hilbert spaces, and such that $T_1 T_2^* T_3 \in E$, $\forall T_1, T_2, T_3 \in E$.

Remark 2.9. It is clear that a ternary ring of operators is also a *-tring; for instance, the *-tring of Example 2.3 is isomorphic to the ternary ring of operators $B(\mathbb{C}^n, \mathbb{C}^m)$.

Definition 2.10. A C^*-norm on a *-ternary ring (E, μ) is a norm which satisfies the following two properties:

1. $\|\mu(x, y, z)\| \leq \|x\| \, \|y\| \, \|z\|$, $\forall x, y, z \in E$.
2. $\|\mu(x, x, x)\| = \|x\|^3$, $\forall x \in E$.

We then say that $(E, \|\cdot\|)$ is a pre-C^*-ternary ring, and if it is a Banach space, we call it a C^*-ternary ring. As in the case of *-trings, we will also use the abbreviated forms pre-C^*-tring and C^*-tring.

Remark 2.11. All the morphisms considered in the context of C^*-trings or ternary rings of operators will be supposed continuous.

Example 2.12. The adjoint *-tring (cf. example 2.5) of a C^*-tring is itself a C^*-tring.

Example 2.13. Suppose $E_1 \subseteq B(H)$, $E_2 \subseteq B(K)$ are closed ternary rings of operators, and consider $E = E_1 \bigoplus E_2$ with the norm $\|(x_1, x_2)\| = \max\{\|x_1\|, \|x_2\|\}$. If $\mu : E \times E \times E \longrightarrow E$ is given by $\mu(\,(x_1, x_2), (y_1, y_2), (z_1, z_2)\,) = (x_1 y_1^* z_1, -x_2 y_2^* z_2)$, then (E, μ) is a C^*-tring.

Remark 2.14. Example 2.13 is in fact the generic example of C^*-tring: by Theorem 3.1 of [11], any C^*-tring $(E,\mu,\|\cdot\|)$ splits uniquely as a direct sum $E = E^+\bigoplus E^-$ of sub-C^*-trings E^+ and E^-, in a way that E^+ is isomorphic to a closed ternary ring of operators, and E^- is anti-isomorphic (2.4) to a closed ternary ring of operators. The sub-C^*-trings E^+ and E^- appear as follows. In Proposition 3.2 of [11], Zettl proves that, up to canonical isomorphisms, there exists a unique pair $(B,\langle\cdot,\cdot\rangle)$ such that:

1. B is a C^*-algebra and E is a right Banach B-module.
2. $\langle\cdot,\cdot\rangle : E\times E\longrightarrow B$ is sesquilinear, conjugate linear in the first variable, with $\|\langle\cdot,\cdot\rangle\|\leq 1$, and
   - $\langle x, yb\rangle = \langle x,y\rangle b$, $\forall x,y\in E$, $b\in B$.
   - $\langle x,y\rangle^* = \langle y,x\rangle$, $\forall x,y\in E$.
3. $\mu(x,y,z) = x\langle y,z\rangle$, $\forall x,y,z\in E$.
4. $\operatorname{span}\langle E,E\rangle$ is dense in B.

Moreover, we have

$$\|x\|^2 = \|\langle x,x\rangle\|, \forall x\in E. \tag{1}$$

Then, he defines $E^+ = \{x\in E : \langle x,x\rangle\in B^+\}$ and $E^- = \{x\in E : \langle x,x\rangle \in -B^+\}$.

Of course, there is also a "left" version of this result, that provides a pair $(A,[\cdot,\cdot])$. However, the sets E^+ and E^- do not depend on which sesquilinear map we are using to define them. In other words, $\langle x,x\rangle\in B^+ \iff [x,x]\in A^+$.

Notation and Terminology

Suppose E is a C^*-tring. When necessary, we will denote the C^*-algebras B and A of Remark 2.14 by B_E and A_E respectively, and the sesquilinear forms $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ by $\langle\cdot,\cdot\rangle_r$ and $\langle\cdot,\cdot\rangle_l$ respectively. We will also use the notation $B'_E := \operatorname{span}\langle E,E\rangle_r$ and $A'_E := \operatorname{span}\langle E,E\rangle_l$. We will refer to the pairs $(B_E,\langle\cdot,\cdot\rangle_r)$ and $(A_E,\langle\cdot,\cdot\rangle_l)$ as the right and left associated pairs of the C^*-tring E respectively. Similar terminology will be used to refer to objects like B_E, B'_E, $\langle\cdot,\cdot\rangle_r$, A_E, A'_E and $\langle\cdot,\cdot\rangle_l$.

With this context in mind, we introduce the next definition.

Definition 2.15. We say that a C^*-tring E is positive if $E = E^+$.

Remark 2.16. Note that, with the notation just introduced, a C^*-tring $(E,\mu,\|\cdot\|)$ becomes a Banach bimodule over $A_E - B_E$, with the additional property that

$$\langle x,y\rangle_l z = \mu(x,y,z) = x\langle y,z\rangle_r,\ \forall x,y,z\in E.$$

In particular, and recalling equation (1) above, we have that if E is a positive C^*-tring, then E becomes a full Hilbert $(A_E - B_E)$-bimodule (by a full bimodule we mean a bimodule that is both full left and full right Hilbert module). We will call it the associated Hilbert bimodule of E (or associated right Hilbert module if we just consider the right action and right inner product). Conversely, a full right Hilbert module $(E,\langle\cdot,\cdot\rangle)$ can be viewed as a positive C^*-tring, by defining $\mu(x,y,z) = x\langle y,z\rangle$ (for a left Hilbert module one defines $\mu(x,y,z) = \langle x,y\rangle z$). On the other hand, we know that a positive C^*-tring is isomorphic to a closed ternary ring of operators (2.14), and it is obvious that any closed ternary ring of operators has a natural structure of positive C^*-tring. In conclusion, positive C^*-trings, full Hilbert modules and closed ternary rings of operators are essentially the same thing, the difference being the adopted point of view in studying them.

We are now prepared to begin our study of tensor products of C^*-trings.

Definition 2.17. A C^*-tensor product of two C^*-trings $(E,\mu,\|\cdot\|)$ and $(F,\nu,\|\cdot\|)$ is a completion of their algebraic tensor product $(E\bigodot F,\mu\odot\nu)$ with respect to a C^*-norm. If γ is such a C^*-norm, we denote by $E\bigotimes_\gamma F$ the corresponding C^*-tensor product.

We will need the following Lemma, whose proof we include here for completeness, although it consists in repeating the proof of Lemma 4.3 of [8], where just the spatial C^*-norm was considered.

Lemma 2.18. *Let A and B be C^*-algebras, and suppose $\mathfrak{a}=\big(a_{ij}\big)$, $\mathfrak{c}=\big(c_{ij}\big)\in M_n(A)$, $\mathfrak{b}=\big(b_{ij}\big)$, $\mathfrak{d}=\big(d_{ij}\big)\in M_n(B)$. Let $A\bigotimes_\gamma B$ be some C^*-tensor product of A and B. Then:*

1. *If $0\leq\mathfrak{a}\leq\mathfrak{c}$ and $0\leq\mathfrak{b}\leq\mathfrak{d}$, we have $\big(a_{ij}\otimes b_{ij}\big)\leq\big(c_{ij}\otimes d_{ij}\big)$ in $M_n(A\bigotimes_\gamma B)$.*
2. *If $\mathfrak{a}$, $\mathfrak{b}\geq 0$, then $\sum_{i,j=1}^n a_{ij}\otimes b_{ij}\geq 0$ in $A\bigotimes_\gamma B$.*

Proof. Let us show first that if $\mathfrak{a}=\big(a_{ij}\big)\in M_n(A)^+$ and $\mathfrak{b}=\big(b_{ij}\big)\in M_n(B)^+$, then $\big(a_{ij}\otimes b_{ij}\big)\in M_n(A\bigotimes_\gamma B)^+$.

Since $\mathfrak{a}=\mathfrak{s}^*\mathfrak{s}$ in $M_n(A)$, and $\mathfrak{b}=\mathfrak{t}^*\mathfrak{t}$ in $M_n(B)$, we will have that, if $\mathfrak{s}=\big(s_{ij}\big)$ and $\mathfrak{t}=\big(t_{ij}\big)$, then

$$a_{ij}=\sum_{k=1}^n s_{ki}^*s_{kj},\quad\text{and}\quad b_{ij}=\sum_{h=1}^n t_{hi}^*t_{hj},\ \forall i,j=1\ldots n.$$

Thus,

$$\big(a_{ij}\otimes b_{ij}\big)=\Big(\sum_{k,h=1}^n s_{ki}^*s_{kj}\otimes t_{hi}^*t_{hj}\Big)=\sum_{h,k=1}^n\big(s_{ki}^*s_{kj}\otimes t_{hi}^*t_{hj}\big).$$

So, it is enough to show that every summand in the last sum is a positive element of $M_n(A\bigotimes_\gamma B)$, and this amounts to prove that if $a_1,\ldots a_n\in A$ and $b_1,\ldots,b_n\in B$, then $\big(a_i^*a_j\otimes b_i^*b_j\big)$ is positive in $M_n(A\bigotimes_\gamma B)$. But $a_i^*a_j\otimes b_i^*b_j=(a_i\otimes b_i)^*(a_j\otimes b_j)$, from which it follows that $\big(a_i^*a_j\otimes b_i^*b_j\big)=\mathfrak{u}^*\mathfrak{u}\in M_n(A\bigotimes_\gamma B)^+$, where $\mathfrak{u}=(u_{ij})$ is the matrix in $M_n(A\bigotimes_\gamma B)$ such that $u_{1i}=a_i\otimes b_i$, and $u_{ij}=0$ for $i>0$.

Now, 1. follows from the fact that $\big((c_{ij}-a_{ij})\otimes b_{ij}\big)$ and $\big(a_{ij}\otimes(d_{ij}-b_{ij})\big)$ are positive elements by the first part of the proof, and then so is their sum.

2. Let $\mathfrak{r}\in M_n(A\bigotimes_\gamma B)$ be such that $(a_{ij}\otimes b_{ij})=\mathfrak{r}^*\mathfrak{r}$. We may suppose $A\bigotimes_\gamma B\subseteq B(H)$, for a Hilbert space H, and therefore $M_n(A\bigotimes_\gamma B)\subseteq B(H^n)$. Consider the operator $\mathfrak{i}:H\longrightarrow H^n$ such that $\mathfrak{i}h=(h,\ldots,h)$, $\forall h\in H$. Then $\sum_{i,j=1}^n a_{ij}\otimes b_{ij}=\mathfrak{i}^*\mathfrak{u}^*\mathfrak{u}\mathfrak{i}\in B(H)^+$. Since $(A\bigotimes_\gamma B)^+=(A\bigotimes_\gamma B)\cap B(H)^+$, we conclude that $\sum_{i,j=1}^n a_{ij}\otimes b_{ij}$ is positive in $A\bigotimes_\gamma B$. $\square$

In the next proposition, we denote the linking algebra of a Hilbert module E by $\mathbb{L}(E)$. Recall that $\mathbb{L}(E)=\begin{pmatrix}A_E & E\\ E^* & B_E\end{pmatrix}$, with the product:

$$\begin{pmatrix}a & e_1\\ e_2 & b\end{pmatrix}\begin{pmatrix}a' & e_1'\\ e_2' & b'\end{pmatrix}=\begin{pmatrix}aa'+\langle e_1,e_2'\rangle_l & ae_1'+e_1b'\\ e_2a'+be_2' & \langle e_2,e_1'\rangle_r+bb'\end{pmatrix}$$

and involution

$$\begin{pmatrix} a & e_1 \\ e_2 & b \end{pmatrix}^* = \begin{pmatrix} a^* & e_2 \\ e_1 & b^* \end{pmatrix}$$

To define the norm on $\mathbb{L}(E)$, consider first the representations

$$\pi_l : \mathbb{L}(E) \longrightarrow \mathfrak{L}(A_E \bigoplus E^*) \text{ such that: } \pi_l \begin{pmatrix} a & e_1 \\ e_2 & b \end{pmatrix} \begin{pmatrix} a' \\ e' \end{pmatrix} = \begin{pmatrix} aa' + \langle e_1, e' \rangle_l \\ e_2 a' + be' \end{pmatrix}$$

$$\pi_r : \mathbb{L}(E) \longrightarrow \mathfrak{L}(E \bigoplus B_E) \text{ such that: } \pi_r \begin{pmatrix} a & e_1 \\ e_2 & b \end{pmatrix} \begin{pmatrix} e' \\ b' \end{pmatrix} = \begin{pmatrix} ae' + e_1 b' \\ \langle e_2, e' \rangle_r + bb' \end{pmatrix}$$

Finally, if $x \in \mathbb{L}(E)$, define

$$\|x\| := \max\{\|\pi_l(x)\|, \|\pi_r(x)\|\}. \tag{2}$$

Consult [1] for more details.

Proposition 2.19. *Let $\mathbb{L}(E)$ and $\mathbb{L}(F)$ be the linking algebras of the associated right Hilbert modules of the C^*-trings E and F respectively (2.16). Then, any C^*-tensor product $\mathbb{L}(E) \bigotimes_\gamma \mathbb{L}(F)$ induces a C^*-tensor product $E \bigotimes_\gamma F$, by restricting the norm γ to the image of the natural inclusion $\iota : E \bigodot F \hookrightarrow \mathbb{L}(E) \bigotimes_\gamma \mathbb{L}(F)$ such that $\sum_i e_i \odot f_i \longmapsto \sum_i \left(\begin{smallmatrix} 0 & e_i \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & f_i \\ 0 & 0 \end{smallmatrix}\right)$.*

Proof. The inclusion $\iota : E \bigodot F \hookrightarrow \mathbb{L}(E) \bigotimes_\gamma \mathbb{L}(F)$ such that $\sum_i e_i \odot f_i \longmapsto \sum_i \left(\begin{smallmatrix} 0 & e_i \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & f_i \\ 0 & 0 \end{smallmatrix}\right)$ is clearly an inclusion of *-ternary rings. So, the restriction of the C^*-norm of $\mathbb{L}(E) \bigotimes_\gamma \mathbb{L}(F)$ to $\iota(E \bigodot F)$ induces a C^*-norm on $E \bigodot F$, as claimed. □

Remark 2.20. Proposition 2.19 implies that if $x = \sum_{i=1}^n e_i \odot f_i \in E \bigodot F$, and $x \neq 0$, then $0 \neq \iota(x)^* \iota(x) = \sum_{i,j=1}^n \langle e_i, e_j \rangle_E \odot \langle f_i, f_j \rangle_F$. Note that this is a very simple way of showing that $\sum_{i,j=1}^n \langle e_i, e_j \rangle_E \odot \langle f_i, f_j \rangle_F \neq 0$, while the proof provided in [8] uses the highly non-trivial stabilization theorem of Kasparov.

Suppose that B_0 is a sub-*-algebra of a C^*-algebra B, and E_0 is a right B_0-module with a definite positive sesquilinear map $\langle \cdot, \cdot \rangle : E_0 \times E_0 \longrightarrow B_0$, that is:

1. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta xz$, $\forall x, y, z \in E_0$, $\alpha, \beta \in \mathbb{C}$.
2. $\langle x, yb \rangle = \langle x, y \rangle b$, $\forall x, y \in E_0$, $b \in B_0$.
3. $\langle y, x \rangle = \langle x, y \rangle^*$, $\forall x, y \in E_0$.
4. $\langle x, x \rangle \geq 0$, $\forall x \in E_0$, and $\langle x, x \rangle = 0 \iff x = 0$.

Then, $\| \cdot \| \longrightarrow \mathbb{R}$ such that $\|x\| = \sqrt{\|\langle x, x \rangle\|}$ is a norm on E_0. Let E be the completion of E_0 with respect to this norm. It is possible to extend the action of B_0 on E_0 to an action of B on E, and also the sesquilinear form to a B-inner product on E, so obtaining a Hilbert B-module. We refer to this construction as the *double completion process* for $(E_0, B_0, \langle \cdot, \cdot \rangle)$ (see [8] for more details).

Proposition 2.21. *Let (E, μ) and (F, ν) be positive C^*-trings with right associated pairs $(B_E, \langle \cdot, \cdot \rangle_E)$ and $(B_F, \langle \cdot, \cdot \rangle_F)$ respectively, and suppose we are given a C^*-norm γ on $B_E \bigodot B_F$.*

1. *Then there is a definite positive sesquilinear map $\langle \cdot, \cdot \rangle (E \bigodot F) \times (E \bigodot F) \longrightarrow B_E \bigodot B_F$ such that $\langle e \odot f, e' \odot f' \rangle = \langle e, e' \rangle_E \odot \langle f, f' \rangle_F$, $\forall e, e' \in E$, $f, f' \in F$.*

2. *The norm* $\tilde{\gamma} : E \bigodot F \longrightarrow \mathbb{R}$ *such that* $\tilde{\gamma}(x) = \sqrt{\gamma(\langle x, x\rangle)}$ *is a* C^**-norm on* $E \bigodot F$*, and the map* $\gamma \longmapsto \tilde{\gamma}$ *is order preserving, from the set* $\mathcal{N}_1$ *of* C^**-norms on* $B_E \bigodot B_F$ *to the set* $\mathcal{N}_2$ *of* C^**-norms on* $E \bigodot F$*.*
3. $E \bigotimes_{\tilde{\gamma}} F$ *is a positive* C^**-tring, with right associated* C^**-algebra* $B_E \bigotimes_{\gamma} B_F$*.*

Proof. Lemma 2.6 shows the existence and uniqueness of such a sesquilinear form $\langle \cdot, \cdot \rangle$. Let us see that $\langle x, x\rangle \geq 0$, $\forall x = \sum_{i=1}^n e_i \odot f_i \in E \bigodot F$:

$$\langle x, x\rangle = \sum_{i,j=1}^n \langle e_i \odot f_i, e_j \odot f_j\rangle = \sum_{i,j=1}^n \langle e_i, e_j\rangle_E \odot \langle f_i, f_j\rangle_F$$

Now, $\left(\langle e_i, e_j\rangle_E\right)_{i,j=1}^n$ and $\left(\langle f_i, f_j\rangle_F\right)_{i,j=1}^n$ are positive elements of $M_n(A)$ and $M_n(B)$ respectively ([8], Lemma 4.2), and therefore $\sum_{i,j=1}^n \langle e_i, e_j\rangle_E \odot \langle f_i, f_j\rangle_F$ is positive in $B_E \bigotimes_\gamma B_F$ by Lemma 2.18. Finally, recall that, by Remark 2.20, $\langle x, x\rangle = 0$ implies $x = 0$. So $\langle \cdot, \cdot\rangle$ is a definite positive sesquilinear form, and then 1. is proved.

By the double completion process described above, we obtain a right Hilbert $B_E \bigotimes_\gamma B_F$-module G, with norm $\tilde{\gamma}$. To prove that $\tilde{\gamma}$ is a C^*-norm on $E \bigodot F$, let us see first that $\mu \odot \nu(x, y, z) = x\langle y, z\rangle$, $\forall x, y, z \in E \bigodot F$; it is clearly enough to show this for $x = e \odot f$, $y = e' \odot f'$, $z = e'' \odot f''$. Now:

$$\begin{aligned}
\mu \odot \nu(x, y, z) &= \mu \odot \nu(e \odot f, e' \odot f', e'' \odot f'') \\
&= \mu(e, e', e'') \odot \nu(f, f', f'') \\
&= (e\langle e', e''\rangle_E) \odot (f\langle f', f''\rangle_F) \\
&= (e \odot f)(\langle e', e''\rangle_E \odot \langle f', f''\rangle_F) \\
&= (e \odot f)\langle e' \odot f', e'' \odot f''\rangle \\
&= x\langle y, z\rangle
\end{aligned}$$

Thus, using the theory of Hilbert modules, we can conclude that $\tilde{\gamma}$ is a C^*-norm:

$$\begin{aligned}
\tilde{\gamma}(\mu \odot \nu(x, y, z)) &= \tilde{\gamma}(x\langle y, z\rangle) \\
&\leq \tilde{\gamma}(x)\gamma(\langle y, z\rangle) \\
&\leq \tilde{\gamma}(x)\tilde{\gamma}(y)\tilde{\gamma}(z)
\end{aligned}$$

And:

$$\begin{aligned}
\tilde{\gamma}(\mu \odot \nu(x, x, x)) &= \tilde{\gamma}(x\langle x, x\rangle) \\
&= \sqrt{\gamma(\langle x\langle x, x\rangle, x\langle x, x\rangle\rangle)} \\
&= \sqrt{\gamma(\langle x, x\rangle^3)} \\
&= \sqrt{\gamma(\langle x, x\rangle)^3} \\
&= \tilde{\gamma}(x)^3
\end{aligned}$$

Moreover, it is obvious that $\gamma \longmapsto \tilde{\gamma}$ is order preserving.

Finally, it follows from 1. and 2., that $E \bigotimes_{\tilde{\gamma}} F$ is a full right Hilbert $B_E \bigotimes_\gamma B_F$-module, and therefore that its associated C^*-algebra is precisely $B_E \bigotimes_\gamma B_F$. □

Next, we want to see that a converse result holds, that is: every C^*-tensor product $E \bigotimes_\alpha F$ induces a tensor product $B_E \bigotimes_{\overline{\alpha}} B_F$ of the corresponding associated C^*-algebras. We need before some preliminary results.

The following Lemma is a slight modification of T.6.1 of [10]. We denote by $M(A)$ the multiplier algebra of the C^*-algebra A and, if I, J are pre-C^*-algebras, and γ is a C^*-norm on the *-algebra $I \bigodot J$, then we indicate its completion by $I \bigotimes_\gamma J$.

Lemma 2.22. *Suppose I, J are two-sided *-ideals (not necessarily closed) of the C^*-algebras A and B respectively, and that γ is a C^*-norm on $I \bigodot J$. Then:*

1. *There exist maps $\iota_A : A \longrightarrow M(I \bigotimes_\gamma J)$ and $\iota_B : B \longrightarrow M(I \bigotimes_\gamma J)$ such that*
$$\iota_A(x)\iota_B(y) = \iota_B(y)\iota_A(x) = x \otimes y, \ \forall x \in I, \, y \in J.$$
2. *If I is dense in A and J is dense in B, ι_A and ι_B are injective.*

Proof. We construct ι_A, the construction of ι_B being analogous. If $\iota : I \bigodot J \longrightarrow I \bigotimes_\gamma J$ is the canonical inclusion, define, for $a \in A$, the maps $L_a, R_a : I \bigodot J \longrightarrow I \bigotimes_\gamma J$ by the formulas $L_a = \iota(l_a \otimes id_J)$, $R_a = \iota(r_a \otimes id_J)$, where $l_a, r_a : I \longrightarrow I$ are the left and right multiplications by the element a respectively; that is, $L_a(x \otimes y) = ax \otimes y$, and $R_a(x \otimes y) = xa \otimes y$. We want to see (L_a, R_a) as a double centralizer of $I \bigotimes_\gamma J$. Note that if $x = \sum_i a_i \otimes b_i$, $y = \sum_j c_j \otimes d_j \in I \bigodot J$, then:
$$\begin{aligned} L_a(xy) &= L_a(x)y \\ R_a(xy) &= xR_a(y) \\ xL_a(y) &= \sum_{i,j} a_i(ac_j) \otimes b_i d_j \\ &= \sum_{i,j} (a_i a)c_j \otimes b_i d_j \\ &= R_a(x)y \end{aligned}$$
If we prove that L_a, R_a are bounded, we will be able to extend them to operators on $I \bigotimes_\gamma J$, obtaining a double centralizer.
So, let $t = \sum_{i=1}^n a_i \otimes b_i \in I \bigodot J$, and consider
$$z = \begin{pmatrix} (\|a\|^2 - a^*a)^{\frac{1}{2}} a_1 \otimes b_1 & \dots & (\|a\|^2 - a^*a)^{\frac{1}{2}} a_n \otimes b_n \\ 0 & \cdots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix} \in M_n(I \bigodot J);$$

Computing, one verifies that $z^*z = (z_{ij})$, where $z_{ij} = a_j^*(\|a\|^2 - a^*a)a_i \otimes b_j^* b_i$. Now, it follows from Lemma 2.18, that:
$$\sum_{i,j=1}^n a_j^*(\|a\|^2 - a^*a)a_i \otimes b_j^* b_i \geq 0$$
and therefore that
$$\sum_{i,j=1}^n a_j^* a^* a a_i \otimes b_j^* b_i \leq \|a\|^2 \sum_{i,j=1}^n a_j^* a_i \otimes b_j^* b_i$$
Hence, $L_a(t)^* L_a(t) \leq \|a\|^2 t^* t$ and, taking norms, we conclude that $\|L_a(t)\| \leq \|a\| \, \|t\|$. So $\|L_a\| \leq \|a\|$, and we can extend $L_a : I \bigotimes_\gamma J \longrightarrow I \bigotimes_\gamma J$. In a similar way, we can see that also $\|R_a\| \leq \|a\|$, and then we have obtained a double centralizer of $I \bigotimes_\gamma J$. It is clear that the map $\iota_A : a \longmapsto (L_a, R_a)$ is a homomorphism of C^*-algebras. It is an easy verification that if ι_B is constructed in the same way, we will have $\iota_A(x)\iota_B(y) = \iota_B(y)\iota_A(x) = x \otimes y$, $\forall x \in I$, $y \in J$. So, 1. is proved.

Let us see now that if I is dense in A, then ι_A is injective: if $(L_a, R_a) = 0$, then $L_a(t) = 0$, $\forall t \in I \bigodot J$, so $ax \otimes y = 0$, $\forall x \in I$, $y \in J$; this implies $ax = 0$, $\forall x \in I$, and then $a = 0$, because I is dense in A. This proves 2. . ∎

Proposition 2.23. *Let I, J be dense two-sided *-ideals of the C^*-algebras A and B respectively. Then every C^*-norm on $I \bigodot J$ may be extended to a C^*norm on $M(A) \bigodot M(B)$.*

Proof. Let γ be a C^*-norm on $I \bigodot J$, and consider the maps $\iota_A : A \longrightarrow M(I \bigotimes_\gamma J)$ and $\iota_B : B \longrightarrow M(I \bigotimes_\gamma J)$ provided by Lemma 2.22. Suppose that $t = \sum_i a_i \otimes b_i \in$ker$(\iota_A \odot \iota_B)$. Since ker$(\iota_A \otimes \iota_B)$ is a two-sided ideal of $A \bigodot B$, we have in particular that $(x \otimes y)t$, $t(x \otimes y) \in$ ker$(\iota_A \odot \iota_B)$, $\forall x \in I$, $y \in J$. This implies $\sum_i xa_i \otimes yb_i = 0 = \sum_i a_i x \otimes b_i y$, $\forall x \in I$, $y \in J$, because $\iota_A \odot \iota_B$ is injective on $I \bigodot J$; and this is also true in any C^*-completion of $A \bigodot B$, for instance in $A \bigotimes_{\min} B$. Since $I \bigodot J$ is dense in $A \bigotimes_{\min} B$, we conclude that $t = 0$. So, $\iota_A \odot \iota_B : A \bigodot B \longrightarrow M(I \bigotimes_\gamma J)$ is injective. Thus, the closure of $\iota_A \odot \iota_B(A \bigodot B)$ in $M(I \bigotimes_\gamma J)$ is a C^*-tensor product of A and B, which has $I \bigodot J$ as a dense subset; we then conclude that it is equal to $I \bigotimes_\gamma J$. The rest of the proof follows now from T.6.3 of [10], which asserts that any C^*-norm on $A \bigodot B$ can be extended to $M(A) \bigodot M(B)$. □

If $(E, \mu, \|\cdot\|)$ is a C^*-tring with right associated pair $(B, \langle\cdot,\cdot\rangle)$, one can consider $\langle x, y\rangle \in B$ as an operator $\theta_{x,y} : E \longrightarrow E$, such that $\theta_{x,y}(z) = \mu(z, x, y) = z\langle x, y\rangle$. Then, one has that $\theta^*_{x,y} = \theta_{y,x}$, and $\|\theta_{x,y}\| = \|\langle x, y\rangle\|$. In the following Proposition we will use this point of view and notation.

Proposition 2.24. *Let $(E, \mu, \|\cdot\|)$ and $(F, \nu, \|\cdot\|)$ be C^*-trings with respective right associated pairs $(B_E, \langle\cdot,\cdot\rangle_E)$ and $(B_F, \langle\cdot,\cdot\rangle_F)$. Suppose that α is a C^*-norm on $E \bigodot F$. Then the right associated C^*-algebra C of the tensor product $E \bigotimes_\alpha F$ is a C^*-completion of $B_E \bigodot B_F$, that is, $B_{E \bigotimes_\alpha F} = B_E \bigotimes_{\overline{\alpha}} B_F$, for a C^*-norm $\overline{\alpha}$ on $B_E \bigodot B_F$.*

Proof. By Proposition 2.23, it is enough to prove that C is a C^*-completion of $B'_E \bigodot B'_F$. Suppose we have such a norm α on $E \bigodot F$. Recall that C is the closure of its two-sided *-ideal C' :=span$\{\theta_{x,x'} : x, x' \in E \bigotimes_\alpha F\}$. Now, if $x = \sum_{i=1}^m e_i \odot f_i \in E \bigodot F$, $x' = \sum_{j=1}^n e'_j \odot f'_j \in E \bigodot F$, $x'' = \sum_{k=1}^n e''_k \odot f''_k \in E \bigodot F$ then, with the obvious notation, we have:

$$\begin{array}{rcl}
\theta_{x,x'}(x'') & = & \eta(x'', x, x') = \mu \odot \nu(x'', x, x') \\
& = & \sum_{i,j,k} \mu(e''_k, e_i, e'_j) \odot \nu(f''_k, f_i, f'_j) \\
& = & \sum_{i,j,k} \theta^E_{e_i,e'_j}(e''_k) \odot \theta^F_{f_i,f'_j}(f''_k) \\
& = & \sum_{i,j,k} (\theta^E_{e_i,e_j} \odot \theta^F_{f_i,f'_j})(e''_k \odot f''_k) \\
& = & \sum_{i,j} (\theta^E_{e_i,e'_j} \odot \theta^F_{f_i,f'_j})(x'')
\end{array}$$

It follows that the function $\theta_{x,x'}$ is an extension to $E \bigotimes_\alpha F$ of the function $\sum_{i,j}(\theta^E_{e_i,e'_j} \odot \theta^F_{f_i,f'_j}) \in B'_E \bigodot B'_F \subseteq L(E \bigodot F)$. This defines a linear map $\phi : B'_E \bigodot B'_F \longrightarrow C$ such that $\phi(\theta^E_{e,e'} \odot \theta^F_{f,f'}) = \theta_{e\odot f, e'\odot f'}$, which is injective, because the image of an element is an extension of this element as a function.

ϕ preserves the involution:

$$\begin{array}{rcl}
\phi(\theta^E_{e,e'} \odot \theta^F_{f,f'})^* & = & (\theta_{e\odot f, e'\odot f'})^* \\
& = & \theta_{e'\odot f', e\odot f} \\
& = & \phi(\theta^E_{e',e} \odot \theta^F_{f',f}) \\
& = & \phi((\theta^E_{e,e'})^* \odot (\theta^F_{f,f'})^*) \\
& = & \phi(\,(\theta^E_{e,e'} \odot \theta^F_{f,f'})^*\,)
\end{array}$$

As for the multiplicativity of ϕ:

$$\begin{aligned}
\phi(\theta^E_{e_1,e_1'}\odot\theta^F_{f_1,f_1'})\phi(\theta^E_{e_2,e_2'}\odot\theta^F_{f_2,f_2'}) &= (\theta_{e_1\odot f_1,e_1'\odot f_1'})(\theta_{e_2\odot f_2,e_2'\odot f_2'})\\
&= \theta_{(e_1\odot f_1,e_2'\odot f_2',e_2\odot f_2),e_1'\odot f_1'}\\
&= \theta_{(e_1,e_2',e_2)\odot(f_1,f_2',f_2),e_1'\odot f_1'}\\
&= \phi(\theta^E_{(e_1,e_2',e_2),e_1'}\odot\theta^F_{(f_1,f_2',f_2),f_1'})\\
&= \phi(\theta^E_{e_1,e_1'}\theta^E_{e_2,e_2'}\odot\theta^F_{f_1,f_1'}\theta^F_{f_2,f_2'})\\
&= \phi\Big((\theta^E_{e_1,e_1'}\odot\theta^F_{f_1,f_1'})(\theta^E_{e_2,e_2'}\odot\theta^F_{f_2,f_2'})\Big)
\end{aligned}$$

Finally, to see that $\phi(B_E'\bigodot B_F')$ is dense in C, note that it contains $\operatorname{span}\{\theta_{x,x'} : x,x'\in E\bigodot F\}$, which is dense in C because $E\bigodot F$ is dense in $E\bigotimes_\alpha F$. □

Theorem 2.25. *Let E and F be positive C^*-trings with right associated C^*-algebras B_E and B_F respectively. Consider the sets*

$$\mathcal{N}_1=\{C^*-\textit{norms on } B_E\bigodot B_F\}$$
$$\mathcal{N}_2=\{C^*-\textit{norms on } E\bigodot F\}$$

and the maps

$$\begin{array}{lll}\phi:\mathcal{N}_1\longrightarrow\mathcal{N}_2: & \gamma\longmapsto\tilde{\gamma} & \textit{defined in Proposition 2.21}\\ \psi:\mathcal{N}_2\longrightarrow\mathcal{N}_1: & \alpha\longmapsto\overline{\alpha} & \textit{defined in Proposition 2.24}\end{array}$$

We consider $\mathcal{N}_1$ and $\mathcal{N}_2$ ordered by the usual order of norms. Then, ϕ and ψ are mutually inverse order preserving bijections.

Proof. Suppose $\gamma\in\mathcal{N}_1$. Then, by 3. of Proposition 2.21, the associated right C^*-algebra of $E\bigotimes_{\tilde{\gamma}}F$ is precisely $B_E\bigotimes_\gamma B_F$; that is: $\overline{\tilde{\gamma}}=\gamma$, and therefore $\psi\phi=id_{\mathcal{N}_1}$.
Consider now $\alpha\in\mathcal{N}_2$. $\overline{\alpha}$ is the C^*-norm on $B_E\bigodot B_F$ such that $B_E\bigotimes_{\overline{\alpha}}B_F=B_{E\bigotimes_\alpha F}$. Now, recall that $\tilde{\overline{\alpha}}:E\bigodot F\longrightarrow\mathbb{R}$ is defined as $\tilde{\overline{\alpha}}(x)=\sqrt{\overline{\alpha}(\langle x,x\rangle)}$, $\forall x\in E\bigodot F$, and $\overline{\alpha}(\langle x,x\rangle)=\alpha(x)^2$ by equation (1) in page 5. So, $\tilde{\overline{\alpha}}=\alpha$, and this shows that $\phi\psi=id_{\mathcal{N}_2}$.
Since ϕ is order preserving by 2. of Proposition 2.21, the proof is complete. □

From now on, we will write just α instead of $\overline{\alpha}$ and γ instead of $\tilde{\gamma}$.

Corollary 2.26. *Let E and F be positive C^*-trings. Then there exist a maximum C^*-norm $\|\cdot\|_{max}$ on $E\bigodot F$, and a minimum C^*-norm on $E\bigodot F$, and $B_{E\bigotimes_{max}F}=B_E\bigotimes_{max}B_F$, $B_{E\bigotimes_{min}F}=B_E\bigotimes_{min}B_F$*

Remark 2.27. Of course, we have left versions of Propositions 2.21 and 2.24, Theorem 2.25 and Corollary 2.26.

As the final result of this section, let us see the following proposition, which will be useful later (see Proposition 4.10). Recall that if E is a Hilbert B-module, $\mathcal{L}(E)=\mathcal{L}_B(E)$ denotes the C^*-algebra of adjointable maps on E, and $\mathcal{K}(E)$ means its sub-C^*-algebra of compact adjointable maps.

Proposition 2.28. *Let E and F be full right Hilbert modules over the C^*-algebras B_E and B_F respectively, and suppose γ is a C^*-norm on $E\bigodot F$. Then γ may be "extended" to a C^*-norm $\hat{\gamma}$ on $\mathcal{L}(E)\bigodot\mathcal{L}(F)$, and we have the following inclusions:*

$$A_E\bigotimes_\gamma A_F\hookrightarrow\mathcal{L}(E)\bigotimes_{\hat{\gamma}}\mathcal{L}(F)\hookrightarrow\mathcal{L}(E\bigotimes_\gamma F),$$

Proof. Observe that with our current notation, $A_E=\mathcal{K}(E)$, $A_F=\mathcal{K}(F)$, and, by the left version of Theorem 2.25, $A_E\bigotimes_\gamma A_F=\mathcal{K}(E\bigotimes_\gamma F)$. On the other hand, one knows that $M(\mathcal{K}(E))=\mathcal{L}(E)$, $M(\mathcal{K}(F))=\mathcal{L}(F)$, and $M(\mathcal{K}(E\bigotimes_\gamma F))=\mathcal{L}(E\bigotimes_\gamma F)$ ([10], Theorem 15.2.12). But then we finish the theorem by recalling that, in general, there exists a C^*-norm $\hat{\gamma}$ such that one has (by Proposition 2.23, for instance):

$$A_E\bigotimes_\gamma A_F\subseteq M(A_E)\bigotimes_{\hat{\gamma}}M(B_F)\subseteq M(A_E\bigotimes_\gamma A_F)$$

$\square$

Remark 2.29. Before ending this section, we would like to remark that Theorem 2.25 is also true without the assumption of positivity on the C^*-trings E and F. In particular, with the obvious definition of nuclear C^*-tring, one can see that the C^*-tring E is nuclear if and only if so is its right associated C^*-algebra. It is also possible to prove that, for a given tensor product $E\bigotimes F$, we have:

$$\begin{aligned}\left(E\bigotimes F\right)^+&=\left(E^+\otimes F^+\right)\bigoplus\left(E^-\otimes F^-\right)\\ \left(E\bigotimes F\right)^-&=\left(E^+\otimes F^-\right)\bigoplus\left(E^-\otimes F^+\right)\end{aligned}$$

3. Tensor Products of Fell Bundles

Our main purpose in the sequel is to introduce a suitable notion of tensor product of Fell bundles (3.5). The tensor product of the Fell bundles $\mathcal{A}=(A_t)_{t\in G}$ and $\mathcal{B}=(B_s)_{s\in H}$ over the discrete groups G and H will be a Fell bundle $\mathcal{C}=(C_r)_{r\in G\times H}$ over $G\times H$, and we will have that C_e is a tensor product of A_e and B_e (we denote by e the unit element of any group). We show that there are, up to isomorphisms, unique tensor products $\mathcal{C}_{\max}$ and $\mathcal{C}_{\min}$ of $\mathcal{A}$ and $\mathcal{B}$, such that $(\mathcal{C}_{\max})_e=A_e\bigotimes_{\max}B_e$ and $(\mathcal{C}_{\min})_e=A_e\bigotimes_{\min}B_e$. Let $\mathcal{A}=(A_t)_{t\in G}$ and $\mathcal{B}=(B_s)_{s\in H}$ be Fell bundles over the groups G and H respectively. Consider, for $t\in G$, $s\in H$, the algebraic tensor product $A_t\bigodot B_s$. Making t, s vary on G and H, we obtain a family $\mathcal{A}\bigodot\mathcal{B}=\{A_t\bigodot B_s\}_{(t,s)\in G\times H}$ of complex vector spaces, indexed by $G\times H$ (we will also see $\mathcal{A}\bigodot\mathcal{B}$ as the disjoint union of the spaces $A_t\bigodot B_s$). Lemma 2.6 shows that, for (t,s), $(t',s')\in G\times H$, we have maps $\left(A_t\bigodot B_s\right)\times\left(A_{t'}\bigodot B_{s'}\right)\longrightarrow A_{tt'}\bigodot B_{ss'}$ such that $(a_t\odot b_s,a_{t'}\odot b_{s'})\longmapsto a_ta_{t'}\odot b_sb_{s'}$, and $A_t\bigodot B_s\longrightarrow A_{t^{-1}}\bigodot B_{s^{-1}}$ such that $a_t\odot b_s\longmapsto a_t^*\odot b_s^*$. Combined, these families of maps produce a product $\cdot:\left(\mathcal{A}\bigodot\mathcal{B}\right)\times\left(\mathcal{A}\bigodot\mathcal{B}\right)\longrightarrow\left(\mathcal{A}\bigodot\mathcal{B}\right)$ and an involution $*:\left(\mathcal{A}\bigodot\mathcal{B}\right)\longrightarrow\left(\mathcal{A}\bigodot\mathcal{B}\right)$ such that the product is associative, bilinear when restricted to each $\left(A_t\bigodot B_s\right)\times\left(A_{t'}\bigodot B_{s'}\right)\longrightarrow A_{tt'}\bigodot B_{ss'}$, $*$ is conjugate linear when restricted to $A_t\bigodot B_s\longrightarrow A_{t^{-1}}\bigodot B_{s^{-1}}$, and $(x\cdot y)^*=y^*\cdot x^*$, $\forall x,y\in\mathcal{A}\bigodot\mathcal{B}$. We call $\mathcal{A}\bigodot\mathcal{B}$ the *algebraic tensor product of* $\mathcal{A}$ *and* $\mathcal{B}$. In other words, $\mathcal{A}\bigodot\mathcal{B}$ is a *-algebraic bundle, in the sense of the following Definition.

Definition 3.1. Let G be a discrete group, and suppose $\mathcal{A}=(A_t)_{t\in G}$ is a family of complex vector spaces. We say that $\mathcal{A}$ is a *-algebraic bundle over G with product $\cdot:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}$ and involution $*:\mathcal{A}\longrightarrow\mathcal{A}$ if, for all $a,b\in\mathcal{A}$, $t,s\in G$:

1. $A_tA_s \subseteq A_{ts}$
2. The product $\cdot$ is bilinear on $A_t \times A_s \longrightarrow A_{ts}$
3. The product on $\mathcal{A}$ is associative.
4. $(A_t)^* \subseteq A_{t^{-1}}$
5. $*$ is conjugate linear from A_t to $A_{t^{-1}}$.
6. $(ab)^* = b^*a^*$.
7. $a^{**} = a$.

   If, in addition, there exists a norm $\|\cdot\| : \mathcal{A} \longrightarrow \mathbb{R}$ such that:
8. $(A_t, \|\cdot\|)$ is a normed space.
9. $\|ab\| \leq \|a\|\,\|b\|$
10. $\|a^*a\| = \|a\|^2$.
11. $a^*a \geq 0$ in the completion of the pre-C^*-algebra $(A_e, \|\cdot\|)$,

we then say that $(\mathcal{A}, \cdot, *, \|\cdot\|)$ is a pre-Fell bundle over the discrete group G, and we refer to $\|\cdot\|$ as a C^*-norm on $\mathcal{A}$. Note that 9. and 10. imply that $\|a^*\| = \|a\|$, $\forall a \in \mathcal{A}$. Recall that if $(A_t, \|\cdot\|)$ is complete for each $t \in G$, then $(\mathcal{A}, \cdot, *, \|\cdot\|)$ is called a Fell bundle.

Proposition 3.2. *Let $\mathcal{A}^0 = (A_t^0)_{t\in G}$ be a pre-Fell bundle over the discrete group G, with C^*-norm $\|\cdot\|$. For each $t \in G$, let A_t be the completion of A_t^0, and consider the family of Banach spaces $(A_t)_{t\in G}$, with the extended norm, that we still denote $\|\cdot\|$. Then the product and involution on $\mathcal{A}^0$ can be extended uniquely to $\mathcal{A}$, in such a way that $\mathcal{A}$ is a Fell bundle over G with the extended maps. We then say that $\mathcal{A}$ is a completion of the pre-Fell bundle $\mathcal{A}^0$.*

Proof. The proof is straightforward, because conditions 9. and 10. of Definition 3.1 ensure that $*$ and $\cdot$ are continuous on $A_t^0 \longrightarrow A_{t^{-1}}^0$ and $A_t^0 \times A_s^0 \longrightarrow A_{ts}^0$ respectively, for all $t, s \in G$, and then they can be extended to continuous maps $A_t \longrightarrow A_{t^{-1}}$ and $A_t \times A_s \longrightarrow A_{ts}$, so defining $\cdot : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $* : \mathcal{A} \longrightarrow \mathcal{A}$. By the continuity of the extended operations, these will satisfy the algebraic conditions 1.–7., and also 8.–11. of 3.1 $\square$

Definition 3.3. Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_t)_{t\in G}$ be *-algebraic bundles over the discrete group G. A morphism $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a function such that $\phi(A_t) \subseteq B_t$, $\forall t \in G$, and, $\forall a, b \in A$, $t \in G$:

1. ϕ is linear on $A_t \longrightarrow B_t$
2. $\phi(ab) = \phi(a)\phi(b)$
3. $\phi(a^*) = \phi(a)^*$

If $\mathcal{A}$, $\mathcal{B}$ are pre-Fell bundles, we also require that ϕ is continuous on each A_t.

Remark 3.4. Note that if $\mathcal{A}$, $\mathcal{B}$ are pre-Fell bundles and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a morphism of *-algebraic bundles, then ϕ is continuous if and only if $\phi : A_e \longrightarrow B_e$ is continuous, because if $x \in \mathcal{A}$, then

$$\|\phi(x)\|^2 = \|\phi(x)^*\phi(x)\| = \|\phi(x^*x)\| \leq \|\phi_{|A_e}\|\,\|x^*x\| = \|\phi_{|A_e}\|\,\|x\|^2$$

In particular, any morphism of *-algebraic bundles between Fell bundles is continuous. Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a morphism between two Fell bundles, and suppose that $C^*(\mathcal{B}) \hookrightarrow B(H)$, as a sub-$C^*$-algebra. The composition of ϕ with this inclusion gives a representation on H of the Fell bundle $\mathcal{A}$. Hence, by the universal property of the cross-sectional C^*-algebra ([7], VIII-17.3), there exists a unique representation $C^*(\phi) : C^*(\mathcal{A}) \longrightarrow B(H)$, that extends ϕ. Since the image

of $\mathcal{A}$ is included in $\mathcal{B} \subseteq C^*(\mathcal{B})$, and since $\mathcal{A}$ is dense in $C^*(\mathcal{A})$, we see that in fact we have $C^*(\phi) : C^*(\mathcal{A}) \longrightarrow C^*(\mathcal{B})$. In conclusion, we have a covariant functor C^* from the category of the Fell bundles over the discrete group G and their morphisms (3.3) to the category of (non-unital) C^*-algebras and their morphisms.

Also note that with the notion of morphisms just introduced, two completions of a given pre-Fell bundle are always isomorphic, and then the completion is essentially unique.

Let us come back to our tensor products.

Definition 3.5. Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles over the discrete groups G and H, and consider their algebraic tensor product $\mathcal{A} \bigodot \mathcal{B}$. If α is a C^*-norm on $\mathcal{A} \bigodot \mathcal{B}$, we call the corresponding completion $\mathcal{A} \bigotimes_\alpha \mathcal{B}$ of $(\mathcal{A} \bigodot \mathcal{B}, \alpha)$ a tensor product of $\mathcal{A}$ and $\mathcal{B}$.

Remark 3.6. Since the main result to be proved in this paper applies just to Fell bundles over discrete groups, we will not discuss the continuous case in what follows. However, let us say that it is not much more difficult to define and construct tensor products of Fell bundles over continuous groups than over discrete groups. In fact, suppose that $\mathcal{A}$ and $\mathcal{B}$ are Fell bundles over the locally compact groups G and H, and $\mathcal{C}$ is a tensor product of $\mathcal{A}$ and $\mathcal{B}$ over the discrete group $G \times H$. If $C_c(\mathcal{A})$ and $C_c(\mathcal{B})$ are the continuous cross-sections of compact support of $\mathcal{A}$ and $\mathcal{B}$ respectively, we define, for $\zeta \in C_c(\mathcal{A})$, $\eta \in C_c(\mathcal{B})$, the map $\zeta \otimes \eta : G \times H \longrightarrow \mathcal{A} \bigodot \mathcal{B} \subseteq \mathcal{C}$ as $\zeta \otimes \eta(t,s) = \zeta(t) \otimes \eta(s)$. Then, using II-13.18 of [7], it is easy to see that $\operatorname{span}\{\zeta \otimes \eta : \ \zeta \in C_c(\mathcal{A}),\ \eta \in C_c(\mathcal{B})\}$ endows $\mathcal{C}$ with a structure of Banach bundle over $G \times H$ and, a posteriori, with a structure of Fell bundle over the locally compact group $G \times H$.

Remark 3.7. If $\mathcal{A} \bigotimes_\alpha \mathcal{B}$ is a tensor product between $\mathcal{A}$ and $\mathcal{B}$, then $\left(\mathcal{A} \bigotimes_\alpha \mathcal{B}\right)_e$ must be a tensor product of A_e and B_e. In fact, if we know the C^*-norm determined by $\left(\mathcal{A} \bigotimes_\alpha \mathcal{B}\right)_e$ on $A_e \bigodot B_e$, then we know the norm of any element $x \in \mathcal{A} \bigotimes_\alpha \mathcal{B}$, because it must be equal to $\sqrt{\alpha(x^*x)}$. Thus, two tensor products will be isomorphic if and only if their unit fibers are the same tensor product of A_e and B_e. A first question that arises is whether every tensor product of A_e and B_e determines a tensor product between the Fell bundles $\mathcal{A}$ and $\mathcal{B}$. Although we will not go deeper in this problem, we will see that this is actually the case for the two tensor product that are really important for our purposes: the minimal and the maximal tensor products (Proposition 3.10).

Proposition 3.8. *Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_t)_{s\in H}$ be Fell bundles, and let α be a C^*-norm on $A_e \bigodot B_e$. Then α can be extended to a C^*-norm (see 3.1) on all of $\mathcal{A} \bigodot \mathcal{B}$ if and only if $\alpha(x^*x) = \alpha(xx^*)$, $\forall x \in \mathcal{A} \bigodot \mathcal{B}$. In this case, the extension is unique.*

Proof. The condition is obviously necessary because of condition 10. of Definition 3.1.

Let us see it is also sufficient. For $x \in \mathcal{A} \bigodot \mathcal{B}$, define $\|x\| = \sqrt{\alpha(x^*x)}$. We want to see that this is a C^*-norm on the *-algebraic bundle $\mathcal{A} \bigodot \mathcal{B}$, that is, it verifies conditions 8.–11. of Definition 3.1. It is clear by the definition of $\|\cdot\|$, that condition 10. is satisfied and, as we are assuming that $\alpha(x^*x) = \alpha(xx^*)$, $\forall x \in \mathcal{A} \bigodot \mathcal{B}$, we also have

$$\|x\| = \|x^*\|,\ \forall x \in \mathcal{A} \bigodot \mathcal{B} \tag{3}$$

Define $\langle\cdot,\cdot\rangle_r : (\mathcal{A}\bigodot\mathcal{B})\times(\mathcal{A}\bigodot\mathcal{B})\longrightarrow\mathcal{A}\bigodot\mathcal{B}$ as $\langle x,y\rangle_r = x^*y$. By 2.21 we know that $\langle\cdot,\cdot\rangle_r$ is an inner product on each $A_t\bigodot B_s$, and it is obvious that $\|x\|^2 = \alpha(\langle x,x\rangle)$, $\forall x\in\mathcal{A}\bigodot\mathcal{B}$. So, conditions 8. and 11. are also satisfied. As for condition 9., observe that the completion of each $A_t\bigodot B_s$ with respect to $\|\cdot\|$ is a right Hilbert $A_e\bigotimes_\alpha B_e$-module; in particular, we have:

$$\|ax\|\leq\|a\|\,\|x\|,\ \forall a\in\mathcal{A}\bigodot\mathcal{B},\, x\in A_e\bigodot B_e \tag{4}$$

Then:

$$\begin{aligned}\|ab\|^2 &= \|b^*a^*ab\| \\ &= \|\langle b,a^*ab\rangle_r\| \\ &\leq \|b\|\,\|a^*ab\| && \text{by Cauchy-Schwarz} \\ &= \|b\|\,\|b^*a^*a\| && \text{by (3)} \\ &\leq \|b\|\,\|b^*\|\,\|a^*a\| && \text{by (4)} \\ &= \|b\|^2\,\|a\|^2 && \text{by (3)}\end{aligned}$$

In conclusion, $\|\cdot\|$ is a C^*-norm on $\mathcal{A}\bigodot\mathcal{B}$.
Finally, the last assertion is due to 3.7. □

Our next purpose is to show that $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ on $A_e\bigodot B_e$ can be extended to $\mathcal{A}\bigodot\mathcal{B}$. We begin with a Lemma that it is possibly a well known fact.

Lemma 3.9. *Let I and J be closed two-sided ideals of the C^*-algebras A and B respectively. Then $I\bigotimes_{max}J$ is the closure of $I\bigodot J$ in $A\bigotimes_{max}B$.*

Proof. Let $\pi : I\bigotimes_{\max}J\longrightarrow B(H)$ be a faithful non-degenerate representation of $I\bigotimes_{\max}J$. Then, it decomposes in faithful non-degenerate representations $\pi_I : I\longrightarrow B(H)$ and $\pi_J : J\longrightarrow B(H)$, such that $\pi_I(x)\pi_J(y) = \pi(x\otimes y) = \pi_J(y)\pi_I(x)$, $\forall x\in I$, $y\in J$. Since they are non-degenerate, there exist unique extensions $\pi_A : A\longrightarrow B(H)$ and $\pi_B : B\longrightarrow B(H)$ to representations of A and B respectively. If $a\in A$, $x\in I$, $b\in B$ and $y\in J$, then $\pi_A(ax)\pi_B(by) = \pi_B(by)\pi_A(ax)$, because $ax\in I$ and $by\in J$. Since π_I and π_J are non-degenerate, we conclude that $\pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$, $\forall a\in A$, $b\in B$. Thus, there exist a representation $\tilde{\pi} : A\bigotimes_{\max}B\longrightarrow B(H)$ such that $\tilde{\pi}(a\otimes b) = \pi_A(a)\pi_B(b)$, $\forall a\in A$, $b\in B$. Then, $\tilde{\pi}$ is an extension of π. Since any representation is norm decreasing, we conclude that if $x\in I\bigodot J$, its norm in $A\bigotimes_{\max}B$ is greater or equal to its maximal norm; so, they are the same. □

Proposition 3.10. *Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles over the discrete groups G and H. Then, the norms $\|\cdot\|_{min}$ and $\|\cdot\|_{max}$ on $A_e\bigodot B_e$ can be extended to C^*-norms on $\mathcal{A}\bigodot\mathcal{B}$. If $\mathcal{A}\bigotimes_\alpha\mathcal{B}$ is a tensor product of $\mathcal{A}$ and $\mathcal{B}$, then there exist canonical surjective morphisms*

$$\mathcal{A}\bigotimes\nolimits_{max}\mathcal{B}\longrightarrow\mathcal{A}\bigotimes\nolimits_{\alpha}\mathcal{B}\longrightarrow\mathcal{A}\bigotimes\nolimits_{min}\mathcal{B}$$

Proof. Let $A_t^*A_t$ be the closure in A_e of $\mathrm{span}\{a_t^*a_t :\ a_t\in A_t\}$, and similarly $B_s^*B_s$. Then $A_t^*A_t$ and $B_s^*B_s$ are closed two-sided ideals of A_e and B_e respectively, and A_t can be seen as a C^*-tring with $A_t^*A_t$ and $A_tA_t^*$ as its right and left associated C^*-algebras respectively, and similarly with B_s. Note that $A_t\bigodot B_s$ is a *-tring, for all $t\in G$, $s\in H$. So, it has a maximum C^*-norm $\|\cdot\|_{\max}$ on it. By Theorem 2.25, its right and left associated C^*-algebras must be $A_t^*A_t\bigotimes_{\max}B_s^*B_s$

and $A_tA_t^* \bigotimes_{\max} B_sB_s^*$ respectively. Since we are dealing with several max norms, let us use the following notation: by $\|\cdot\|_{\max}$, $\|\cdot\|_{\max}^r$, $\|\cdot\|_{\max}^l$ and $\|\cdot\|_\mu$ we mean respectively the maximum norms on $A_t \bigodot B_s$, $A_t^*A_t \bigodot B_s^*B_s$, $A_tA_t^* \bigodot B_sB_s^*$, and $A_e \bigodot B_e$. Since every $A_t \bigotimes_{\max} B_s$ is a Hilbert $\big(A_tA_t^* \bigotimes_{\max} B_sB_s^*–A_t^*A_t \bigotimes_{\max} B_s^*B_s\big)$-bimodule, we know that

$$\|xx^*\|_{\max}^l = \|x\|_{\max}^2 = \|x^*x\|_{\max}^r, \ \forall x \in A_t \bigodot B_s.$$

Finally, by Lemma 3.9, $\|\cdot\|_\mu$ coincides with $\|\cdot\|_{\max}^r$ on $A_t^*A_t \bigodot B_s^*B_s$ and with $\|\cdot\|_{\max}^l$ on $A_tA_t^* \bigodot B_sB_s^*$. We deduce that

$$\|xx^*\|_\mu = \|x\|_{\max}^2 = \|x^*x\|_\mu, \ \forall x \in \mathcal{A} \bigodot \mathcal{B}.$$

So, by Proposition 3.8, $\|\cdot\|_{\max}$ on $A_e \bigodot B_e$ can be extended to all of $\mathcal{A} \bigodot \mathcal{B}$. The same arguments show that $\|\cdot\|_{\min}$ can be extended, because the minimum norm is the spatial one, and, by the construction of the spatial tensor product, it is clear that the restriction of the $\|\cdot\|_{\min}$ on $A_e \bigodot B_e$ to any $I \bigodot J$ gives the spatial norm of $I \bigodot J$, where I and J are sub-C^*-algebras of A_e and B_e respectively.

The proof of the last assertion is trivial: the morphisms are the continuous extensions of the identity, that we know is continuous on $A_e \bigodot B_e$, and therefore on all the bundle (see Remark 3.4). □

4. Cross–sectional Algebras of Tensor Products of Fell Bundles

In this section we study some connections between amenability of Fell bundles and tensor products of Fell bundles over discrete groups. The main result, Theorem 4.11, provides a commutative diagram that relates, in particular, the C^*-algebras $C^*(\mathcal{A} \bigotimes_{\max} \mathcal{B})$, $C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B})$, $C_r^*(\mathcal{A} \bigotimes_{\min} \mathcal{B})$ and $C_r^*(\mathcal{A}) \bigotimes_{\min} C_r^*(\mathcal{B})$, and from which we easily obtain many Corollaries, among them the result announced in the abstract.

Recall from [4] that, in addition to the cross-sectional C^*-algebra $C^*(\mathcal{B})$ of the Fell bundle $\mathcal{B}$ over a discrete group G, there is another C^*-algebra naturally associated to $\mathcal{B}$: the reduced cross-sectional C^*-algebra $C_r^*(\mathcal{B})$. Briefly, this C^*-algebra is obtained as follows: for each $t \in G$, $b_t \in B_t$, one defines an operator $\Lambda(b_t) : \ell^2(\mathcal{B}) \longrightarrow \ell^2(\mathcal{B})$ as $\Lambda(b_t)\xi_{|_s} = b_t\xi(t^{-1}s)$, where

$$\ell^2(\mathcal{B}) = \{\xi : G \longrightarrow \mathcal{B}/\xi(t) \in B_t, \forall t \in G, \text{ and } \sum_{t\in G} \xi(t)^*\xi(t) \text{ converges in norm in } B_e\}$$

With $\langle\cdot,\cdot\rangle : \ell^2(\mathcal{B}) \times \ell^2(\mathcal{B}) \longrightarrow B_e$ given by $\langle\xi,\eta\rangle = \sum_{t\in G} \xi(t)^*\eta(t)$, $\ell^2(\mathcal{B})$ is a full right Hilbert B_e-module, and $\Lambda(b_t)$ is adjointable ($\Lambda(b_t)^* = \Lambda(b_t^*)$). Finally, $C_r^*(\mathcal{B})$ is the sub-C^*-algebra of $\mathcal{L}_{B_e}\big(\ell^2(\mathcal{B})\big)$ generated by the set $\{\Lambda(b_t) :\ t \in G,\ b_t \in B_t\}$. Λ is called the left regular representation of $\mathcal{B}$, and can be thought as a map $\Lambda : C^*(\mathcal{B}) \longrightarrow C_r^*(\mathcal{B})$. Since Λ is always an epimorphism, it will be an isomorphism if and only if $\ker\Lambda = 0$. In this case, $C^*(\mathcal{B}) = C_r^*(\mathcal{B})$, and we say that $\mathcal{B}$ is amenable. Related to the amenability of Fell bundles is the notion of a Fell bundle with the approximation property ([4], 4.4):

Definition 4.1. A Fell bundle $\mathcal{B} = (B_t)_{t\in G}$ over a discrete group G has the approximation property if there exists a net $(a_i)_{i\in I}$ of finitely supported functions $a_i : G \longrightarrow B_e$ such that:

1. It is uniformly bounded, that is: $\sup_{i\in I} \|\sum_{t\in G} a_i(t)^*a_i(t)\| = M < \infty$.
2. For each $t \in G$ and $b_t \in B_t$ we have: $\lim_i \sum_{r\in G} a_i(tr)^*b_ta_i(r) = b_t$.

We will refer to such a net as an “approximating net”.

The relevance of this concept is that a Fell bundle that has the approximation property is automatically amenable, that is, its reduced cross-sectional C^*-algebra agrees with the full one ([4], Theorem 4.6).

Our first result shows that the class of Fell bundles with the approximation property is closed under tensor products.

Proposition 4.2. *If $\mathcal{A} = (A_t)_{t\in G}$, $\mathcal{B} = (B_s)_{s\in H}$ are Fell bundles over the discrete groups G and H respectively, and they have the approximation property, then any tensor product $\mathcal{C} = \mathcal{A} \bigotimes \mathcal{B}$ also has the approximation property. In particular, $\mathcal{A} \bigotimes \mathcal{B}$ is amenable.*

Proof. Let $(\alpha_i)_{i\in I}$ and $(\beta_j)_{j\in J}$ be approximating nets of finitely supported functions corresponding to $\mathcal{A}$ and $\mathcal{B}$ respectively, with

$$\sup_{i\in I} \| \textstyle\sum_{r\in G} \alpha_i(r)^*\alpha_i(r)\| = M < \infty, \ \sup_{j\in J} \| \textstyle\sum_{s\in H} \beta_j(s)^*\beta_j(s)\| = N < \infty.$$

Consider, for every $(i, j) \in I \times J$, the function $\gamma_{i,j} : G \times H \longrightarrow A_e \bigotimes B_e$, given by $\gamma_{i,j} = \alpha_i \otimes \beta_j$. We will show that $(\gamma_{i,j})_{(i,j)\in I\times J}$ is an approximating net for $\mathcal{C}$, where we consider the product order on $I \times J$: $(i, j) \geq (i', j')$ if and only if $i \geq i'$, $j \geq j'$.

First, note that every $\gamma_{i,j}$ has finite support. Also, the net is uniformly bounded, i.e.:

$$\begin{aligned} \| \textstyle\sum_{(r,s)\in G\times H)} \gamma_{i,j}(r,s)^*\gamma_{i,j}(r,s)\| &= \| \textstyle\sum_{(r,s)\in G\times H} (\alpha_i(r)^* \otimes \beta_j(s)^*)(\alpha_i(r) \otimes \beta_j(s))\| \\ &= \| \textstyle\sum_{r\in G, s\in H} \alpha_i(r)^*\alpha_i(r) \otimes \beta_j(s)^*\beta_j(s)\| \\ &= \|(\textstyle\sum_{r\in G} \alpha_i(r)^*\alpha_i(r)) \otimes (\textstyle\sum_{s\in H} \beta_j(s)^*\beta(s))\| \\ &= \| \textstyle\sum_{r\in G} \alpha_i(r)^*\alpha_i(r)\| \, \| \textstyle\sum_{s\in H} \beta_j(s)^*\beta_j(s)\| \\ &\leq MN. \end{aligned}$$

Finally, we should show that if $(g, h) \in G \times H$ and $x \in A_g \bigotimes B_h$, we have:

$$\lim_{(i,j)\in I\times J} \sum_{(r,s)\in G\times H} \gamma_{i,j}((g,h)(r,s))^* x \gamma_{ij}(r,s) = x$$

Consider the linear map

$$T_{i,j} : A_g \bigotimes B_h \longrightarrow A_g \bigotimes B_h$$

such that

$$T_{i,j}(x) = \sum_{(r,s)\in G\times H} \gamma_{i,j}((g,h)(r,s))^* x \gamma_{i,j}(r,s)$$

This is a bounded map, with norm at most MN (1.12 of [5]).

In the case that $x = a_g \otimes b_h$, we have:

$$\begin{aligned} T_{i,j}(a_g \otimes b_h) &= \sum_{(r,s)\in G\times H} (\alpha_i(gr)^* \otimes \beta_j(hs)^*)(a_g \otimes b_h)(\alpha_i(r) \otimes \beta_j(s)) \\ &= \sum_{(r,s)\in G\times H} (\alpha_i(gr)^* a_g \alpha_i(r)) \otimes (\beta_j(hs)^* b_s \beta_j(s)) \\ &= \big(\sum_{r\in G} \alpha_i(gr)^* a_g \alpha_i(r)\big) \otimes \big(\sum_{s\in H} \beta_j(hs)^* b_h \beta_j(s)\big) \end{aligned}$$

So, $\lim_{(i,j)} T_{i,j}(a_g \otimes b_h) = a_g \otimes b_h$.
Next, if $x \in A_g \bigotimes B_h$, given $\epsilon > 0$, let $y \in A_g \bigodot B_h$ be such that $\|x - y\| \leq \frac{\epsilon}{2(MN+1)}$. The computations above show that there exists $(i_0, j_0) \in I \times J$ such that $\|T_{i,j}(y) - y\| < \frac{\epsilon}{2}$, $\forall (i,j) \geq (i_0, j_0)$. Then, if $(i,j) \geq (i_0, j_0)$:

$$\begin{aligned}\|T_{i,j}(x) - x\| &\leq \|T_{i,j}(x-y)\| + \|T_{i,j}(y) - y\| + \|y - x\| \\ &\leq (MN+1)\|x - y\| + \|T_{i,j}(y) - y\| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

We conclude that $\lim_{(i,j)} T_{i,j}(x) = x$, $\forall x \in A_g \bigotimes B_h$, which finishes the proof. □

Let $\mathcal{B} = (B_t)_{t \in G}$ be a Fell bundle over the discrete group G. If $X \subseteq G$, by $\mathcal{B}_X$ we mean the reduced Fell bundle $\mathcal{B}_X = (B_t)_{t \in \langle X \rangle}$, where $\langle X \rangle$ denotes the subgroup of G generated by the set X. Before the next proposition, let us recall the notion of topological grading of a C^*-algebra over a discrete group ([4], Definitions 3.1 and 3.4).

Definition 4.3. Let B be a C^*-algebra and suppose that for each element t of a discrete group G we are given a closed linear subspace $B_t \subseteq B$. The family $(B_t)_{t \in G}$ is called a *grading of* B if, for all s, $t \in G$, we have:

1. $B_t B_s \subseteq B_{ts}$.
2. $B_t^* = B_{t^{-1}}$
3. The closed linear span of $\bigcup_{t \in G} B_t$ is dense in B.
4. The spaces B_t are linearly independent.

If, in addition, there is a bounded linear map $F : B \longrightarrow B_e$ such that F is the identity map on B_e and that F vanishes on each B_t, for $t \neq e$, we say that the family $(B_t)_{t \in G}$ is a *topological grading of the C^*-algebra B*.

Remark 4.4. If the family $(B_t)_{t \in G}$ satisfies properties 1., 2., 3., and there exists a map $F : B \longrightarrow B_e$ as in 4.3, then $(B_t)_{t \in G}$ is a topological grading of the C^*-algebra B, by Theorem 3.3 of [4], where it is also shown that F is positive, contractive, and a conditional expectation, and if $\mathcal{B}$ denotes the associated Fell bundle, then there exist canonical C^*-algebra epimorphisms $\rho : C^*(\mathcal{B}) \longrightarrow B$ and $\lambda : B \longrightarrow C_r^*(\mathcal{B})$. One always has that $(B_t)_{t \in G}$ is a topological grading of $C^*(\mathcal{B})$ and of $C_r^*(\mathcal{B})$.

The next proposition could be useful in some situations (see for instance Corollary 4.15). Recall that $C_c(\mathcal{B})$ is the set of continuous sections of compact support of the Fell bundle $\mathcal{B}$ (in our case compact is the same of finite, because we are considering just discrete groups).

Proposition 4.5. *Let $\mathcal{B} = (B_t)_{t \in G}$ be a Fell bundle over the discrete group G, and suppose $(G_i)_{i \in I}$ is a family of subgroups of G such that $G = \bigcup_{i \in I} G_i$. Call $\mathcal{B}_i = \mathcal{B}_{G_i}$. If $\mathcal{B}_i$ is amenable for every i, then so is $\mathcal{B}$, and $\varinjlim C^*(\mathcal{B}_i) = C^*(\mathcal{B})$.*

Proof. We have the following inclusions:

$$C_c(\mathcal{B}_i) \hookrightarrow C_c(\mathcal{B}) \hookrightarrow C^*(\mathcal{B}),$$

and

$$C_c(\mathcal{B}_i) \hookrightarrow C_c(\mathcal{B}) \hookrightarrow C^*_r(\mathcal{B}).$$

Completing $C_c(\mathcal{B})$ with respect to each of these inclusions, we obtain C^*-algebras C_i and D_i respectively. Observe that $(B_t)_{t\in G_i}$ is a topological grading of both C_i and D_i, because restricting to C_i and to D_i the corresponding conditional expectations of $C^*(\mathcal{B})$ and $C^*_r(\mathcal{B})$, one falls in the conditions of 4.3. Using now that $\mathcal{B}_i$ is amenable, we conclude that $C_i = D_i = C^*(\mathcal{B}_i)$. In this way, we obtain inclusions

$$C^*(\mathcal{B}_i) \hookrightarrow C^*(\mathcal{B}) \quad \text{and} \quad C^*(\mathcal{B}_i) \hookrightarrow C^*_r(\mathcal{B}).$$

Since $C_c(\mathcal{B}) = \bigcup_{i\in I} C_c(\mathcal{B}_i)$ is dense in both $C^*(\mathcal{B})$ and $C^*_r(\mathcal{B})$, and we have:

$$C_c(\mathcal{B}) \subseteq \bigcup_{i\in I} C^*(\mathcal{B}_i) \subseteq C^*(\mathcal{B})$$

and

$$C_c(\mathcal{B}) \subseteq \bigcup_{i\in I} C^*(\mathcal{B}_i) \subseteq C^*_r(\mathcal{B}),$$

we conclude that

$$C^*_r(\mathcal{B}) = \varinjlim C^*(\mathcal{B}_i) = C^*(\mathcal{B}),$$

as we wanted. □

With the next Proposition, we begin the proof of the main result (Theorem 4.11). From now on, we will use the following notation: if $\rho : X \longrightarrow G$ is a surjective function, where for each $t \in G$ the set $\rho^{-1}(t)$ is a vector space, then for any $x \in X$ we denote by $\zeta_x : G \longrightarrow X$ the function such that $\zeta_x(t) = 0_t$ if $t \neq \rho(x)$, and $\zeta_x(t) = x$ if $\rho(x) = t$; other letters, like ξ or η, will also be used (here 0_t stands for the zero vector of $\rho^{-1}(t)$). In the same context, the notation x_t for $x_t \in X$ means that $\rho(x_t) = t$.

Proposition 4.6. *Let $\mathcal{A} = (A_t)_{t\in G}$, $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles over the discrete groups G and H respectively. Then:*

$$C^*(\mathcal{A} \bigotimes\nolimits_{max} \mathcal{B}) \cong C^*(\mathcal{A}) \bigotimes\nolimits_{max} C^*(\mathcal{B})$$

Proof. We have the canonical inclusions and isomorphism:

$$A_g \bigodot B_h \hookrightarrow \bigoplus_{t,s} \big(A_t \bigodot B_s\big) \cong \big(\textstyle\bigoplus_t A_t\big) \bigodot \big(\textstyle\bigoplus_s B_s\big) \hookrightarrow C^*(\mathcal{A}) \bigodot C^*(\mathcal{B})$$

So, we get an inclusion $A_g \bigodot B_h \hookrightarrow C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B})$, and it is clear that this is an inclusion of *-trings. Now, since we are considering the maximum norm on $A_g \bigodot B_h$, this inclusion will be continuous, and hence it can be extended to all of $A_g \bigotimes_{\max} B_h$ by continuity. The collection of maps $\{A_g \bigotimes_{\max} B_h \longrightarrow C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B})\}_{(g,h)\in G\times H}$ obtained is a representation of $\mathcal{A} \bigotimes_{\max} \mathcal{B}$ (at least if we consider $C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B})$ as faithfully represented on some Hilbert space), and thus induces an epimorphism

$$\Phi : C^*(\mathcal{A} \bigotimes\nolimits_{\max} \mathcal{B}) \longrightarrow C^*(\mathcal{A}) \bigotimes\nolimits_{\max} C^*(\mathcal{B}),$$

which is the identity on $\bigoplus_{(t,s)\in G\times H} (A_t \bigodot B_s)$ (up to the natural isomorphism $\bigoplus_{t,s} \big(A_t \bigodot B_s\big) \cong \big(\bigoplus_t A_t\big) \bigodot \big(\bigoplus_s B_s\big)$ mentioned above).

Suppose now that $C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B})$ is faithfully represented in $B(H)$ as a non-degenerated C^*-algebra, for some Hilbert space H. This gives a non-degenerate and injective representation of $\mathcal{A}\bigotimes_{\max}\mathcal{B}$. Let $(e_i)_{i\in I}$ be an approximate unit of A_e. If we set $\hat{e_i}=\zeta_{e_i}$, i.e.:

$$\hat{e_i}: G\longrightarrow\mathcal{A} \text{ such that : } \hat{e_i}(t)=\begin{cases} e_i & \text{if } t=e,\\ 0 & \text{otherwise}\end{cases},$$

we know that $(\hat{e_i})_{i\in I}$ is an approximate unit for $C^*(\mathcal{A})$ (by a direct check or using [7] VIII-5.11 and VIII-16.3). Define:

$$\phi_i:\ell^1(\mathcal{B})\longrightarrow\ell^1(\mathcal{A}\bigotimes_{\max}\mathcal{B}) \text{ such that: } \phi_i(\eta)(r,s)=\hat{e_i}(r)\otimes\xi(s)$$

Then ϕ_i is continuous, because:

$$\|\phi_1(\xi)\|_1=\sum_{(r,s)\in G\times H}\|\hat{e_i}(r)\otimes\xi(s)\|=\sum_{s\in H}\|e_i\|\,\|\xi(s)\|=\|e_i\|\,\|\xi\|_1.$$

Since the inclusion $\ell^1(\mathcal{A}\bigotimes_{\max}\mathcal{B})\hookrightarrow C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B})$ is continuous, we can think ϕ_i as a continuous, in fact, contractive map $\phi_i:\ell^1(\mathcal{B})\longrightarrow B(H)$. Similarly, for an approximate unit $(f_j)_{j\in J}$ of B_e, we have an approximate unit $(\hat{f}_j)_{j\in J}$ of $C^*(\mathcal{B})$ and maps $\psi_j:\ell^1(\mathcal{B})\longrightarrow B(H)$.

Next, we want to see that the nets $\big(\phi_i(\xi)\big)_{i\in I}$ and $\big(\psi_j(\eta)\big)_{j\in J}$ converge strongly to operators $\phi(\xi)$ and $\psi(\eta)$. If $\zeta\in\ell^1(\mathcal{A}\bigotimes_{\max}\mathcal{B})$,

$$\begin{aligned}\phi_i(\xi)\zeta(r',s') &= \sum_{(r,s)\in G\times H}\phi_i(\xi)(r,s)\zeta(\,(r,s)^{-1}(r',s')\,)\\ &= \sum_{(r,s)\in G\times H}\big(\hat{e_i}(r)\otimes\xi(s)\big)\zeta(r^{-1}r',s^{-1}s')\\ &= \sum_{s\in H}\big(e_i\otimes\xi(s)\big)\zeta(r',s^{-1}s')\end{aligned}$$

In particular we have (see the comment on notation just before this Proposition):

$$\begin{aligned}\phi_i(\xi)\zeta_{a_{r_1}\otimes b_{s_1}}(r,s) &= \sum_{u\in H}\big(e_i\otimes\xi(u)\big)\zeta_{a_{r_1}\otimes b_{s_1}}(r,u^{-1}s)\\ &= \begin{cases} e_ia_{r_1}\otimes\xi(ss_1^{-1})b_{s_1} & \text{if } r=r_1\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

(in the special case of $\xi=\hat{f}_j$, we obtain $\phi_i(\hat{f}_j)\zeta_{a_r\otimes b_s}=\zeta_{e_ia_r\otimes f_jb_s}$). Then, $\phi_i(\xi)\zeta_{a_{r_1}\otimes b_{s_1}}=\sum_{s\in H}\zeta_{e_ia_{r_1}\otimes\xi(ss_1^{-1})b_{s_1}}$ in $\ell^1(\mathcal{A}\bigotimes_{\max}\mathcal{B})$. Since $C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B})\subseteq B(H)$ is non-degenerated, the set $\mathcal{S}=\operatorname{span}\{\zeta_{a_r\otimes b_s}(h):\ a_r\in A_r,\ b_s\in B_s,\ h\in H\}$ is dense in H. Now, given $\zeta_{a_r\otimes b_s}(h)\in\mathcal{S}$:

$$\begin{aligned}\phi_i(\xi)\big(\zeta_{a_r\otimes b_s}(h)\big) &= \big(\phi_i(\xi)\zeta_{a_r\otimes b_s}\big)(h)\\ &= \sum_{u\in H}\zeta_{e_ia_r\otimes\xi(us^{-1})b_s}(h)\end{aligned}$$

To see that this is convergent, it is enough to show that $\lim_i \phi_i(\xi)\zeta_{a_r\otimes b_s} = \sum_{u\in H}\zeta_{a_r\otimes\xi(us^{-1})b_s}$ in $\ell^1(\mathcal{A}\bigotimes_{\max}\mathcal{B})$. Now:

$$\begin{aligned}\|\textstyle\sum_{u\in H}\zeta_{e_ia_r\otimes\xi(us^{-1})b_s}-\sum_{u\in H}\zeta_{a_r\otimes\xi(us^{-1})b_s}\|_1 &= \sum_{(r',s')\in G\times H}\|\textstyle\sum_{u\in H}\zeta_{e_ia_r-a_r\otimes\xi(us^{-1})b_s}(r',s')\|\\ &= \sum_{u\in H}\|\big(e_ia_r-a_r\big)\otimes\xi(us^{-1})b_s\|\\ &\le \|e_ia_r-a_r\|\sum_{u\in H}\|\xi(us^{-1})b_s\|\\ &\le \|e_ia_r-a_r\|\,\|b_s\|\,\|\xi_1\|\\ &\longrightarrow 0 \text{ as } i\longrightarrow\infty\end{aligned}$$

Since the set $\{\phi_i\}$ is bounded, and there exists $\lim_i \phi_i(\xi)(h)$, $\forall h\in\mathcal{S}$, we conclude that there exists $\lim_i \phi_i(\xi)(h) =: \phi(\xi)(h)$, $\forall h\in H$. This defines a contraction $\phi:\ell^1(\mathcal{B})\longrightarrow B(H)$, and it is easy to verify that ϕ is in fact a representation of $\ell^1(\mathcal{B})$:

$$\begin{aligned}\phi(\xi)\phi(\eta)\zeta_{a_r\otimes b_s} &= \phi(\xi)\sum_{u\in H}\zeta_{a_r\otimes\eta(us^{-1})b_s}\\ &= \sum_{u\in H}\phi(\xi)\zeta_{a_r\otimes\eta(us^{-1})b_s}\\ &= \sum_{u\in H}\sum_{v\in H}\zeta_{a_r\otimes\xi(vu^{-1})\eta(us^{-1})b_s}\\ &= \sum_{w\in H}\sum_{v\in H}\zeta_{a_r\otimes\xi(w)\eta(w^{-1}vs^{-1})b_s}\\ &= \sum_{v\in H}\zeta_{a_r\otimes(\xi\eta)(vs^{-1})b_s}\\ &= \phi(\xi\eta)\zeta_{a_r\otimes b_s}\end{aligned}$$

and:

$$\begin{aligned}\langle\phi(\xi)\zeta_{a_r\otimes b_s}(h),\zeta_{a_{r'}\otimes b_{s'}}(h')\rangle &= \langle\textstyle\sum_{u\in H}\zeta_{a_r\otimes\xi(us^{-1})b_s}(h),\zeta_{a_{r'}\otimes b_{s'}}(h')\rangle\\ &= \langle h,\big(\textstyle\sum_{u\in H}\zeta_{a_r\otimes\xi(us^{-1})b_s}\big)^*\zeta_{a_{r'}\otimes b_{s'}}(h')\rangle\\ &= \langle h,\textstyle\sum_{u\in H}\zeta_{a_r^*\otimes b_s^*\xi(us^{-1})^*}\zeta_{a_{r'}\otimes b_{s'}}(h')\rangle\\ &= \langle h,\textstyle\sum_{u\in H}\zeta_{a_r^*a_{r'}\otimes b_s^*\xi(us^{-1})^*b_{s'}}(h')\rangle\\ &= \langle\zeta_{a_r\otimes b_s}(h),\textstyle\sum_{u\in H}\zeta_{a_{r'}\otimes\xi(us^{-1})^*b_{s'}}(h')\rangle\\ &= \langle\zeta_{a_r\otimes b_s}(h),\textstyle\sum_{u\in H}\zeta_{a_{r'}\otimes\xi(us^{-1}s'(s')^{-1})b_{s'}}(h')\rangle\\ &= \langle\zeta_{a_r\otimes b_s}(h),\textstyle\sum_{w\in H}\zeta_{a_{r'}\otimes\xi(w(s')^{-1})b_{s'}}(h')\rangle\\ &= \langle\zeta_{a_r\otimes b_s}(h),\phi(\xi^*)\zeta_{a_{r'}\otimes b_{s'}}(h')\rangle\end{aligned}$$

Then, we can extend ϕ to $C^*(\mathcal{B})$. In the same way we obtain a representation $\psi:\ell^1(\mathcal{A})\longrightarrow B(H)$ as a strong limit of the ψ_i's, and we extend it to a representation of $C^*(\mathcal{A})$. It is clear that ϕ and ψ are commuting representations. Therefore, by the universal property of the maximal tensor product, we get a morphism

$$\phi\otimes\psi: C^*(\mathcal{A})\bigotimes_{\max}C^*(\mathcal{B})\longrightarrow B(H).$$

Observe that in fact the image of this morphism is exactly $C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B})$, because this is closed in $B(H)$, $\phi\otimes\psi\big(C_c(\mathcal{A})\bigodot C_c(\mathcal{B})\big)\subseteq\ell^1(\mathcal{A}\bigotimes_{\max}\mathcal{B})$, and $\overline{\ell^1(\mathcal{A}\bigotimes_{\max}\mathcal{B})} = C^*(\mathcal{A})\bigotimes_{\max}C^*(\mathcal{B})$.

Finally, we have that the composition

$$(\phi\otimes\psi)\Phi : C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B})\longrightarrow C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B})$$

is the canonical epimorphism, which in this case is the identity. With the same argument, we also conclude that $\Phi(\phi\otimes\psi)$ is the identity on $C^*(\mathcal{A})\bigotimes_{\max} C^*(\mathcal{B})$. Thus, Φ is actually an isomorphism. □

Proposition 4.7. *Let $\mathcal{A}$, $\mathcal{B}$ be Fell bundles over G and H respectively, and suppose α, β are C^*-norms on $\mathcal{A}\bigodot\mathcal{B}$, with $\alpha\geq\beta$. Then, there is a canonical epimorphism*

$$\Theta : C^*(\mathcal{A}\bigotimes_{\alpha}\mathcal{B})\longrightarrow C^*(\mathcal{A}\bigotimes_{\beta}\mathcal{B}),$$

which is the identity on $C_c(\mathcal{A}\bigodot\mathcal{B})$.

Proof. For the existence of Θ see Remark 3.4; Θ is an epimorphism because $C_c(\mathcal{A}\bigodot\mathcal{B})$ is dense in $C^*(\mathcal{A}\bigotimes_{\beta}\mathcal{B})$. □

Lemma 4.8. *Let $\mathcal{A}$, $\mathcal{B}$ be Fell bundles over G and H respectively, and suppose α is a C^*-norm on $\mathcal{A}\bigodot\mathcal{B}$. Then*

$$\ell^2(\mathcal{A})\bigotimes_{\alpha}\ell^2(\mathcal{B})\cong\ell^2(\mathcal{A}\bigotimes_{\alpha}\mathcal{B})$$

Proof. Given the C^*-norm α on $\mathcal{A}\bigodot\mathcal{B}$, we know from Theorem 2.25 that there exists exactly one Hilbert $A_e\bigotimes_{\alpha}B_e$-module having $\ell^2(\mathcal{A})\bigodot\ell^2(\mathcal{B})$ as a dense sub-*-tring. So, all what we should show is that $\ell^2(\mathcal{A})\bigodot\ell^2(\mathcal{B})$ is included as a dense sub-*-tring of the full Hilbert $A_e\bigotimes_{\alpha}B_e$-module $\ell^2(\mathcal{A}\bigotimes_{\alpha}\mathcal{B})$. Using again the isomorphism $\big(\bigoplus_{t\in G}A_t\big)\bigodot\big(\bigoplus_{s\in H}B_s\big)\cong\bigoplus_{(t,s)\in G\times H}\big(A_t\bigodot B_s\big)$, and the fact that these subspaces are dense in both of the compared Hilbert $A_e\bigotimes_{\alpha}B_e$-modules, it suffices to prove that the corresponding inner products agree on this set. For this, it is enough to do the comparison on a generating set. The set $\{\xi_{a_t}\odot\xi_{b_s}/\ a_t\in\mathcal{A},\ b_s\in\mathcal{B}\}$ generates $\big(\bigoplus_{t\in G}A_t\big)\bigodot\big(\bigoplus_{s\in H}B_s\big)$. Note that, when viewed as an element of $\bigoplus_{(t,s)\in G\times H}\big(A_t\bigodot B_s\big)$, $\xi_{a_t}\odot\xi_{b_s}$ is precisely the element $\xi_{a_t\odot b_s}$; we will use the different notations to indicate the space where we consider the element. Now:

$$\begin{aligned}\langle\xi_{a_t}\odot\xi_{b_s},\xi_{a'_{t'}}\odot\xi_{b'_{s'}}\rangle &= \langle\xi_{a_t},\xi_{a'_{t'}}\rangle\odot\langle\xi_{b_s},\xi_{b'_{s'}}\rangle\\ &= \big(\sum_{u\in G}\xi_{a_t}(u)^*\xi_{a'_{t'}}(u)\big)\odot\big(\sum_{v\in H}\xi_{b_s}(v)^*\xi_{b'_{s'}}(v)\big)\\ &= \begin{cases}0 & \text{if } t\neq t' \text{ or } s\neq s'\\ \xi_{a_t^*a_t'\odot b_s^*b_s'} & \text{otherwise}\end{cases}\\ &= \langle\xi_{a_t\odot b_s},\xi_{a'_{t'}\odot b'_{s'}}\rangle\end{aligned}$$

and this is all we needed. □

Note that we have natural inclusions $A_e\hookrightarrow C^*_r(\mathcal{A})\subseteq\mathcal{L}(\ell^2(\mathcal{A}))$ and $B_e\hookrightarrow C^*_r(\mathcal{B})\subseteq\mathcal{L}(\ell^2(\mathcal{B}))$. So, $A_e\bigodot B_e\subseteq C^*_r(\mathcal{A})\bigodot C^*_r(\mathcal{B})\subseteq\mathcal{L}(\ell^2(\mathcal{A}))\bigodot\mathcal{L}(\ell^2(\mathcal{B}))$.

Definition 4.9. Let α be a C^*-norm on $\mathcal{A} \bigodot \mathcal{B}$. By Proposition 2.28, α induces a C^*-norm $\hat{\alpha}$ on $\mathcal{L}(\ell^2(\mathcal{A})) \bigodot \mathcal{L}(\ell^2(\mathcal{B}))$. We define $C_r^*(\mathcal{A}) \bigotimes_{\hat{\alpha}} C_r^*(\mathcal{B})$ as the closure of $C_r^*(\mathcal{A}) \bigodot C_r^*(\mathcal{B})$ in $\mathcal{L}(\ell^2(\mathcal{A})) \bigotimes_{\hat{\alpha}} \mathcal{L}(\ell^2(\mathcal{B}))$.

Proposition 4.10. *Let $\mathcal{A}$, $\mathcal{B}$ be Fell bundles over G and H respectively, and let α be a C^*-norm on $\mathcal{A} \bigodot \mathcal{B}$. Then*

$$C_r^*(\mathcal{A}) \bigotimes_{\hat{\alpha}} C_r^*(\mathcal{B}) \cong C_r^*(\mathcal{A} \bigotimes_{\alpha} \mathcal{B})$$

Proof. We have seen in Proposition 2.28, that we have an embedding

$$\mathcal{L}(\ell^2(\mathcal{A})) \bigotimes_{\hat{\alpha}} \mathcal{L}(\ell^2(\mathcal{B})) \hookrightarrow \mathcal{L}(\ell^2(\mathcal{A}) \bigotimes_{\alpha} \ell^2(\mathcal{B}))) \cong \mathcal{L}(\mathcal{A} \bigotimes_{\alpha} \mathcal{B})$$

Indeed, the definition of $\mathcal{L}(\ell^2(\mathcal{A})) \bigotimes_{\hat{\alpha}} \mathcal{L}(\ell^2(\mathcal{B}))$ is precisely the closure of $\mathcal{L}(\ell^2(\mathcal{A})) \bigodot \mathcal{L}(\ell^2(\mathcal{B}))$ in $\mathcal{L}(\ell^2(\mathcal{A}) \bigotimes_{\alpha} \ell^2(\mathcal{B}))$. This implies that $C_r^*(\mathcal{A}) \bigotimes_{\hat{\alpha}} C_r^*(\mathcal{B}) \hookrightarrow \mathcal{L}(\ell^2(\mathcal{A} \bigotimes_{\alpha} \mathcal{B}))$. In order to prove the proposition, it is enough to see that some dense subset of $C_r^*(\mathcal{A}) \bigotimes_{\hat{\alpha}} C_r^*(\mathcal{B})$ is mapped onto a dense subset of $C_r^*\big(\ell^2(\mathcal{A} \bigotimes_{\alpha} \mathcal{B})\big)$. Let us find the image of the elements like $\Lambda(a_t) \otimes \Lambda(b_s)$, where $a_t \in A_t$, $b_s \in B_s$. It suffices to know how $\Lambda(a_t) \otimes \Lambda(b_s)$ acts on elements of the form $\xi \otimes \eta$, where $\xi \in C_c(\mathcal{A})$, $\eta \in C_c(\mathcal{B})$:

$$\begin{aligned}
\Lambda(a_t) \otimes \Lambda(b_s)(\xi \otimes \eta)_{|_{(x,y)}} &= \Lambda(a_t)\xi_{|_x} \otimes \Lambda(b_s)\eta_{|_y} \\
&= a_t\xi(t^{-1}x) \otimes b_s\eta(s^{-1}y) \\
&= (a_t \otimes b_s)\xi \otimes \eta\big((t,s)^{-1}(x,y)\big) \\
&= \Lambda(a_t \otimes b_s)(\xi \otimes \eta)_{|_{(x,y)}}
\end{aligned}$$

Thus, $\Lambda(a_t \otimes b_s)$ is the continuous extension of $\Lambda(a_t) \otimes \Lambda(b_s)$ to all of $\ell^2(\mathcal{A}) \bigotimes_{\alpha} \ell^2(\mathcal{B}) \cong \ell^2(\mathcal{A} \bigotimes_{\alpha} \mathcal{B})$. This finishes the proof, because $\operatorname{span}\{\Lambda(a_t) \otimes \Lambda(b_s) : \ a_t \in A_t,\, b_s \in B_s\}$ is dense in $C_r^*(\mathcal{A}) \bigotimes_{\alpha} C_r^*(\mathcal{B})$, and $\operatorname{span}\{\Lambda(a_t \otimes b_s) : \ a_t \in A_t,\, b_s \in B_s\}$ is dense in $C_r^*\big(\ell^2(\mathcal{A} \bigotimes_{\hat{\alpha}} \mathcal{B})\big)$ ◻

Theorem 4.11. *Let $\mathcal{A} = (A_t)_{t\in G}$, $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles over the discrete groups G and H respectively. Then, for every C^*-norm α on $\mathcal{A} \bigodot \mathcal{B}$, we have the following commutative diagram:*

$$D_\alpha : \qquad \begin{tikzcd}
C^*(\mathcal{A} \bigotimes_{max} \mathcal{B}) \arrow[r, two heads, "\Theta_\alpha"] \arrow[d, "\Phi\cong"'] & C^*(\mathcal{A} \bigotimes_{\alpha} \mathcal{B}) \arrow[r, two heads, "\Lambda_\alpha"] & C_r^*(\mathcal{A} \bigotimes_{\alpha} \mathcal{B}) \\
C^*(\mathcal{A}) \bigotimes_{max} C^*(\mathcal{B}) \arrow[rr, two heads, "\Lambda_{\mathcal{A}} \otimes \Lambda_{\mathcal{B}}"'] & & C_r^*(\mathcal{A}) \bigotimes_{\hat{\alpha}} C_r^*(\mathcal{B}) \arrow[u, "\cong\Psi"']
\end{tikzcd}$$

where Θ_α is the epimorphism provided by Proposition 4.7, Φ and Ψ the isomorphisms of Proposition 4.6 and of Proposition 4.10 respectively, and Λ_α, $\Lambda_{\mathcal{A}}$ and $\Lambda_{\mathcal{B}}$ are the left regular representations.

Proof. There is nothing to be proved: any of the morphisms appearing in the diagram is the identity on $C_c(\mathcal{A} \bigodot \mathcal{B}) \cong C_c(\mathcal{A}) \bigodot C_c(\mathcal{B})$. ◻

Corollary 4.12. *The Fell bundle $\mathcal{A} \bigotimes_{max} \mathcal{B}$ is amenable if and only if $\mathcal{A}$, $\mathcal{B}$ are amenable Fell bundles and $C^*(\mathcal{A}) \bigotimes_{max} C^*(\mathcal{B}) = C^*(\mathcal{A}) \bigotimes_{\widehat{max}} C^*(\mathcal{B})$.*

Proof. For $\alpha = \max$, the corresponding diagram $D_{\max}$ becomes:

$$\begin{tikzcd} C^*(\mathcal{A} \bigotimes_{\max} \mathcal{B}) \arrow[r, two heads, "\Lambda"] \arrow[d, "\Phi\cong"'] & C_r^*(\mathcal{A} \bigotimes_{\max} \mathcal{B}) \\ C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B}) \arrow[r, two heads, "\Lambda_{\mathcal{A}} \otimes \Lambda_{\mathcal{B}}"'] & C_r^*(\mathcal{A}) \bigotimes_{\widehat{\max}} C_r^*(\mathcal{B}) \arrow[u, "\cong\Psi"'] \end{tikzcd}$$

So Λ is an isomorphism if and only if $\Lambda_{\mathcal{A}} \otimes \Lambda_{\mathcal{B}}$ is, that is, $C^*(\mathcal{A} \bigotimes_{\max} \mathcal{B}) = C_r^*(\mathcal{A} \bigotimes_{\max} \mathcal{B})$ if and only if $C^*(\mathcal{A}) = C_r^*(\mathcal{A})$, $C^*(\mathcal{B}) = C_r^*(\mathcal{B})$, and $\|\cdot\|_{\max} = \|\cdot\|_{\widehat{\max}}$. $\square$

For the next corollary, note that $C_r^*(\mathcal{A}) \bigotimes_{\widehat{\min}} C_r^*(\mathcal{B})= C_r^*(\mathcal{A}) \bigotimes_{\min} C_r^*(\mathcal{B})$

Corollary 4.13. *If $\mathcal{A}$ and $\mathcal{B}$ are Fell bundles with the approximation property, then $A_e \bigodot B_e$ admits exactly one C^*-norm if and only if $C^*(\mathcal{A}) \bigodot C^*(\mathcal{B})$ admits exactly one C^*-norm. In particular, if $\mathcal{A}$ is a Fell bundle with the approximation property whose unit fiber A_e is nuclear, then $C^*(\mathcal{A})$ is also nuclear.*

Proof. Since $\mathcal{A}$, $\mathcal{B}$ are Fell bundles with the approximation property, then $\mathcal{A} \bigotimes_{\min} \mathcal{B}$ also has the approximation property, so the diagram $D_{\min}$ becomes:

$$\begin{tikzcd} C^*(\mathcal{A} \bigotimes_{\max} \mathcal{B}) \arrow[r, two heads, "\Theta_{\min}"] \arrow[d, "\Phi\cong"'] & C^*(\mathcal{A} \bigotimes_{\min} \mathcal{B}) \\ C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B}) \arrow[r, two heads, "\Lambda_{\mathcal{A}} \otimes \Lambda_{\mathcal{B}}"'] & C^*(\mathcal{A}) \bigotimes_{\min} C^*(\mathcal{B}) \arrow[u, "\cong\Psi"] \end{tikzcd}$$

If $A_e \bigodot B_e$ admits just one C^*-tensor norm, then $\mathcal{A} \bigotimes_{\max} \mathcal{B} = \mathcal{A} \bigotimes_{\min} \mathcal{B}$ and this implies that $\Theta_{\min} = id$, and hence that $\Lambda_{\mathcal{A}} \otimes \Lambda_{\mathcal{B}}$ also is an isomorphism; therefore $C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B}) = C^*(\mathcal{A}) \bigotimes_{\min} C^*(\mathcal{B})$. Conversely, suppose that $C^*(\mathcal{A}) \bigodot C^*(\mathcal{B})$ admits just one C^*-norm. Then $\Lambda_{\mathcal{A}} \otimes \Lambda_{\mathcal{B}}$ is an isomorphism, which shows that $\Theta_{\min}$ is also an isomorphism. This implies that $\mathcal{A} \bigotimes_{\max} \mathcal{B}=\mathcal{A} \bigotimes_{\min} \mathcal{B}$, and therefore that $A_e \bigotimes_{\max} B_e=A_e \bigotimes_{\min} B_e$. As for the last assertion, observe that any C^*-algebra B may be regarded as a Fell bundle over the trivial group, and it is clear that any approximate unit of the algebra will do the service of an approximating net for this Fell bundle. Now, by the first part of the corollary, one concludes that $C^*(\mathcal{A}) \bigotimes_{\max} B = C^*(\mathcal{A}) \bigotimes_{\min} B$, i.e., $C^*(\mathcal{A})$ is a nuclear C^*-algebra. $\square$

Corollary 4.14. *If $\mathcal{A}$ is a Fell bundle with the approximation property and nuclear unit fiber, and if $\mathcal{B}$ is an amenable Fell bundle, then $\mathcal{A} \bigotimes \mathcal{B}$ is amenable.*

Proof. By Corolary 4.13, the assumptions on $\mathcal{A}$ imply that $C^*(\mathcal{A})$ is nuclear. Thus, it must be $C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B})= C^*(\mathcal{A}) \bigotimes_{\widehat{\max}} C^*(\mathcal{B})$ and the result follows from Corollary 4.12. $\square$

Corollary 4.15. *Suppose that the Fell bundle $\mathcal{B}$ is in the hypothesis of Proposition 4.5. If B_e is nuclear and each $\mathcal{B}_i$ has the approximation property, then $C^*(\mathcal{B})$ is nuclear.*

Proof. Since B_e is nuclear and each $\mathcal{B}_i$ has the approximation property, by 4.13 we have that $C^*(\mathcal{B}_i)$ is nuclear, $\forall i \in I$, and hence $\varinjlim C^*(\mathcal{B}_i)$ is also nuclear; but $C^*(\mathcal{B}) = \varinjlim C^*(\mathcal{B}_i)$ by Proposition 4.5. $\square$

Corollary 4.16. *Any twisted partial crossed product of a nuclear C^*-algebra by an amenable discrete group is nuclear. In particular, the partial C^*-algebra $C_p^*(G)$ of an amenable discrete group is nuclear.*

Proof. Recall from [3] that a twisted partial crossed product of a locally compact group gives rise to a Fell bundle over the group. Now, if the group is discrete and amenable, the Theorem 4.7 of [4] shows that the Fell bundle has the approximation property, and this is all we need, because of Corollary 4.13. The last assertion is true because $C_p^*(G)$ is by definition a partial crossed product of a commutative C^*-algebra by the group G (see [2], Defitnition 6.4). $\square$

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